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THE N/G/1 QUEUE AND ITS DETAILED ANALYSIS.(U)

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6 THE N/G/1 QUEUE AND ITS DETAILED ANALYSIS.

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ABSTRACT

We discuss a single server queue whose input is the versatile Markovian point process recently introduced by M. F. Neuts (c.f. Tech Report #77/13, NR Dept. of Statistics & CS, Univ. of Delaware), herein to be called the N-Process. Special cases of the N-Process discussed earlier in the literature include a number of complex models such as the Markov-modulated Poisson Process, the superposition of a Poisson Process and a Phase Type Renewal Process etc. This queueing model has great appeal in its applicability to real world situations especially such as those involving inhibition or stimulation of arrivals by certain renewals. The paper presents formulas in forms which are computationally tractable and provides a unified treatment of many models which were discussed earlier by several authors and which turn out to be special cases. Among the topics discussed are busy period characteristics, queue length distributions, moments of the queue length and virtual waiting time. The analysis presented here serves as an example of the power of Markov Renewal Theory.

CHAPTER I
THE N-PROCESS AND THE N/G/1 QUEUE

1.1 Introduction

In this Chapter we discuss a versatile class of point processes on the real line which are closely related to finite-state Markov processes and indicate how a substantial number of models hitherto used in the literature for modeling arrival processes are particular cases. This process was recently introduced by M. F. Neuts [22], and we shall henceforth refer to it as an N-Process. Herein we also define the N/G/1 queue which is the subject matter of this paper.

Section 1.2 discusses probability distributions of Phase Type (PH-Distributions) and Phase Type Renewal Processes (PH-Renewal Processes) which were also introduced by Neuts [14,20] and which form the sub-strata for the definition of the N-Process. In Section 1.3 we define the N-Process and summarize some useful results regarding such a process.

In Section 1.4 we define the N/G/1 queueing model and describe the semi-Markov sequence imbedded therein. Finally, the last section provides a number of interesting special cases of the N/G/1 model some of which have been previously discussed.

1.2 Phase Type Distributions and Phase Type Renewal Processes

Although the phase concept has been used extensively in the literature since its introduction by A. K. Erlang [5], the use of general phase distributions has remained limited until recently. The simplest distributions of phase type due to Erlang and bearing his name have been generalized by some authors [3,9] by considering mixtures of them, different parameters for different phases, random number of phases etc. A systematic discussion of the general phase concept and the accruing benefits in modeling a wide variety of interesting qualitative features especially of interest in Queueing Theory are due to Neuts [14] who has also demonstrated the power of the method of phases in a series of papers [15,16,17]. We refer the reader to the cited references for a complete discussion of Phase Type distributions and their usefulness, giving only a summary of those results pertinent to our discussion.

Consider a continuous-time Markov Process with state-space $\{1,\ldots,m,m+1\}$ for which the states $1,\ldots,m$ are transient and the state $m+1$ is absorbing. We assume that starting at any transient state, absorption into $m+1$ is almost certain. The infinitesimal generator Q of such a Markov Process then has the form

$$Q = \begin{bmatrix} T & \underline{T}^{\circ} \\ \underline{0} & 0 \end{bmatrix}, \qquad (1.2.1)$$

where T is an $m\times m$ matrix with $T_{ii}<0$ and $T_{ij}\geq 0$ for $i\neq j$ such that T^{-1} exists. The vector $\underline{T}°$ is nonnegative and satisfies $T\underline{e}+\underline{T}°=\underline{0}$, where $\underline{e}=(1,\ldots,1)'$. A vector $(\underline{\alpha}, \alpha_{m+1})$ of initial probabilities is also given and satisfies $\underline{\alpha}\underline{e}+\alpha_{m+1}=1$, $0\leqq\alpha_{m+1}<1$.

For the above Markov Process, the probability distribution $F(\cdot)$ of the time till absorption in the state (m+1) is given by

$$F(x) = 1-\underline{\alpha}\exp(Tx)\underline{e}, \quad x\geqq 0 \tag{1.2.2}$$

<u>Definition 1.2.3</u>: Any probability distribution $F(\cdot)$ on $[0,\infty)$ constructed as above will be called a <u>Phase Type Distribution</u> (PH-Distribution). The pair $(\underline{\alpha},T)$ will be called <u>a representation</u> of $F(\cdot)$.

In the sequel we shall assume that $\alpha_{m+1}=0$ so that $F(\cdot)$ does not have an atom at 0. In [14] it is shown that one may, without loss of generality, assume that the representation $(\underline{\alpha},T)$ of $F(\cdot)$ is so chosen that the matrix

$$Q^* = T+T°A°, \tag{1.2.4}$$

where $T°$ is an m×m matrix all whose columns are $\underline{T}°$ and $A°=\mathrm{diag}(\alpha_1,\ldots,\alpha_m)$, is irreducible. Henceforth we assume that this is indeed the case.

The matrix Q^*, which is of considerable importance, is the infinitesimal generator of the <u>Phase Type Renewal Process (PH-Renewal Process)</u> which is obtained by restarting the Markov Process Q instantaneously after each absorption (renewal) by performing a multinomial trial with probabilities $\underline{\alpha}$ and outcomes 1,...,m. Note that the times between successive renewals of such a renewal process is the

PH-Distribution $F(\cdot)$ described above thereby suggesting its nomenclature.

For later use we also introduce the following notations. $\underline{\theta}$ will denote the invariant probability vector of the Markov Process Q^*, i.e., the unique (strictly positive) vector satisfying

$$\underline{\theta} Q^* = \underline{0}, \quad \underline{\theta}\underline{e} = 1. \tag{1.2.5}$$

We recall from [20] that the stationary version of the PH-Renewal Process is obtained by starting the Markov Process Q^* with initial probability vector $\underline{\theta}$. We also recall that the mean of $F(\cdot)$ is given by

$$\mu_1' = -\underline{\alpha} T^{-1} \underline{e}. \tag{1.2.6}$$

It is now easily verified that

$$\underline{\theta} = \frac{1}{\mu_1'} (-\underline{\alpha} T^{-1}). \tag{1.2.7}$$

In the sequel $A^{\circ\circ}$ will denote an $m \times m$ matrix all whose rows are $\underline{\alpha}$. Also the $m \times m$ matrix each of whose rows is $\underline{\theta}$ will be denoted by Θ.

1.3 The N-Process

The Markov Process Q^* described in Section 1.2 will be the sub-stratum for the definition of the N-Process. A transition in the Markov Process Q^* from the state i to the state j will be called an <u>(i,j)-transition</u> if it does not involve a renewal (i.e., no visit to the "instantaneous" state $(m+1)$), and an <u>(i,j)-renewal transition</u> otherwise. Note that unlike the former, the latter may go from a state to itself. We are now ready to describe the arrival process of interest in terms of the following assumptions.

Assumptions regarding arrival epochs and group sizes

(A) During any sojourn of the Markov Process Q* in the state i, $1 \leqq i \leqq m$, there are Poisson arrivals of rate λ_i and group size density $\{p_i(k): k \geqq 0\}$. We let $\phi_i(z)$ denote the p.g.f. of $\{p_i(k)\}$ and let $\underline{\phi}(z)=(\phi_1(z),\ldots,\phi_m(z))$.

(B) At (i,j)-renewal transitions there are group arrivals with probability density $\{r_{ij}(k): k \geqq 0\}$ whose p.g.f. is $\Phi_{ij}(z)$. Let $\Phi(z)$ denote the $m \times m$ matrix of entries $\Phi_{ij}(z)$.

(C) At (i,j)-transitions, $i \neq j$, there are group arrivals with probability density $\{q_{ij}(k): k \geqq 0\}$ whose p.g.f. is $\psi_{ij}(z)$. For notational convenience in the sequel we set $\psi_{ii}(z) \equiv 1$, $1 \leqq i \leqq m$ and let $\psi(z)$ denote the $m \times m$ matrix of entries $\psi_{ij}(z)$.

Independence Assumptions

(D) For every $t>0$, given the path function of the Markov Process Q* the epochs of the Type A arrivals are conditionally independent given the successive sojourn times, and behave as a homogeneous Poisson process on every sojourn interval.

(E) Given the times and types of the arrival epochs up to time t, the group sizes are conditionally independent and have the probability densities given above.

Definition 1.3.1: The arrival process defined by the foregoing assumptions (A)-(E) is called an N-Process.

Let $N(t)$ and $J(t)$, $t \geqq 0$, denote respectively the number of arrivals in $(0,t]$ and the state of the Markov Process Q* at t+. (J(t) will be referred to as the <u>phase</u> at t). It is then easy to see that $\{(N(t),J(t)): t \geq 0\}$ is a Markov Process with state-space $\{0,1,\ldots\} \times \{1,\ldots,m\}$.

In [22] it is shown that the m×m matrices of probabilities $P(\nu,t)=(P_{ij}(\nu,t))$, $\nu \geq 0$, $t \geq 0$, where

$$P_{ij}(\nu,t)=P\{N(t)=\nu, J(t)=j \mid N(0)=0, J(0)=i\}, \tag{1.3.2}$$

have generating function

$$\tilde{P}(z,t)=\sum_{\nu=0}^{\infty} z^{\nu} P(\nu,t)=\exp[R(z)t], \quad |z| \leq 1, \tag{1.3.3}$$

with

$$R(z)=\Delta(\underline{\lambda})\Delta(\underline{\phi}(z))-\Delta(\underline{\lambda})+T\circ\psi(z)+T^{\circ}A^{\circ}\circ\Phi(z), \tag{1.3.4}$$

where

$$\Delta(\underline{\lambda})=\mathrm{diag}(\lambda_1,\ldots,\lambda_m), \tag{1.3.5}$$

and

$$\Delta(\underline{\phi}(z))=\mathrm{diag}(\phi_1(z),\ldots,\phi_m(z)), \tag{1.3.6}$$

and '$\circ$' here and in the sequel denotes the Schur product (entrywise product) of two matrices. Further it is shown that the matrix

$$M(t) = \left[\frac{\partial}{\partial z} \tilde{P}(z,t)\right]_{z=1-} \tag{1.3.7}$$

is given by

$$M(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\nu=0}^{n-1} Q^{*\nu} \; R'(1) \; Q^{*n-1-\nu}, \tag{1.3.8}$$

where

$$R'(1) = \Delta(\underline{\lambda}\circ\underline{\gamma})+T\circ C+T^{\circ}A^{\circ}\circ D, \tag{1.3.9}$$

with

$$\underline{\gamma}=\underline{\phi}'(1-), \quad C=\psi'(1-), \quad D=\Phi'(1-) \tag{1.3.10}$$

$$\Delta(\underline{\lambda}\circ\underline{\gamma}) = \mathrm{diag}(\lambda_1\gamma_1,\ldots,\lambda_m\gamma_m). \tag{1.3.11}$$

Also the vector

$$\underline{\xi}(t) = M(t)\underline{e}, \tag{1.3.12}$$

whose j-th component is the expected number of arrivals in (0,t] given J(0)=j is given by

$$\underline{\xi}(t)=\xi^* t\underline{e}+(I-\Theta)(\tau^*\Theta-Q^*)^{-1}R'(1)\underline{e} + [\Theta-\exp(Q^*t)](\tau^*\Theta-Q^*)^{-1}R'(1)\underline{e}, \tag{1.3.13}$$

where

$$\xi^* = \underline{\theta}R'(1)\underline{e} \tag{1.3.14}$$

and τ^* is any real number such that $\tau^* \geq \max_i(-Q^*_{ii})$. We also recall that ξ^* is the arrival rate for the stationary version of *the N-Process which is obtained by starting the underlying* Markov Process Q^* according to $\underline{\theta}$.

The following theorem gives an interesting interpretation for the quantity ξ^* which will be useful later.

Theorem 1.3.15: ξ^* is the ratio of the expected number of arrivals during a typical renewal interval of the underlying PH-renewal process to the expected length of that renewal interval.

Proof: Let N_1, N_2, N_3 denote respectively the number of arrivals of types (A), (B) and (C) (described in the definition of the N-Process) during a typical renewal interval. It is easily seen that

$$E(N_1) = \sum_j \int_{0-}^{\infty} [\underline{\alpha} \exp(Tx)]_j \lambda_j \gamma_j dx = -\underline{\alpha} T^{-1} \Delta(\underline{\lambda} \circ \underline{\gamma}) \underline{e},$$

$$E(N_2) = \sum_j \sum_k \int_{0-}^{\infty} [\underline{\alpha} \exp(Tx)]_j T^{\circ}_j \alpha_k D_{jk} dx = -\underline{\alpha} T^{-1} (T^{\circ} A^{\circ} \circ D) \underline{e},$$

and

$$E(N_3) = \sum_j \sum_k \int_{0-}^{\infty} [\underline{\alpha} \exp(Tx)]_j T_{jk} C_{jk} dx = -\underline{\alpha} T^{-1} (T \circ C) \underline{e}.$$

Adding the above three quantities and dividing by μ_1', we have, by (1.2.6), (1.2.7) and (1.3.9) that

$$\frac{1}{\mu_1'} E(N_1 + N_2 + N_3) = \underline{\theta} R'(1) \underline{e} = \xi^*,$$

and hence the result.

Remark: In view of the above theorem we may consider ξ^* as a "generalized" arrival rate for the N-Process.

We now present some results concerning the matrix

$$R(0) = \Delta(\underline{\lambda}) \Delta(\underline{\phi}(0)) - \Delta(\underline{\lambda}) + T \circ \psi(0) + T^{\circ} A^{\circ} \circ \Phi(0)$$

which are somewhat technical in nature and which will be found necessary for the sequel. Before we do this, let us recall [12] the following regarding Stability matrices.

Definition 1.3.16: An $m \times m$ matrix A of complex numbers is said to be semi-stable if $\mathrm{Re}(\delta_i) \leqq 0$ for every eigenvalue δ_i of A. It is stable if $\mathrm{Re}(\delta_i) < 0$ for every i.

Lemma 1.3.17: [12]: If $A = (a_{ij})$ is an $m \times m$ real matrix, $a_{ij} \geqq 0$ for $i \neq j$, and there exist positive numbers $t_1, \ldots, t_m$ such that

$$\sum_j t_j a_{ij} \leqq 0, \quad i = 1, \ldots, m,$$

then A is semi-stable.

We are now ready to prove

<u>Theorem 1.3.18</u>: The matrix $R(0)$ is semi-stable.

<u>Proof</u>: By comparing $R(0)$ with Q^* one can easily show that for all i, $R_{ii}(0)<0$ and $\sum_j R_{ij}(0)\leqq 0$. Since $R_{ij}(0)\geqq 0$ for all $i\neq j$ as is seen directly, $R(0)$ satisfies the conditions of Lemma 1.3.17 with $t_1=\ldots=t_m=1$.

Before we proceed with our discussion of $R(0)$ we list the following well-known results governing a nonnegative $m\times m$ matrix $A\neq 0$. We refer to Gantmacher [6] for the proofs of these results.

(R1): $|\delta_i| \leqq \max_i \sum_j a_{ij}$ for every eigenvalue δ_i of A.

(R2): There exists a nonnegative eigenvalue δ of A satisfying $\delta\geqq|\delta_i|$ for any other eigenvalue δ_i of A. δ is called the Perron-Frobenius (PF) eigenvalue of A.

(R3): Suppose B is irreducible and $B\geqq A$. If the PF-eigenvalues of A and B are equal, then $A=B$.

(R4): If A is stochastic, then the PF-eigenvalue of A is 1.

Suppose now that the matrix $R(0)$ is not stable. Then by Theorem 1.3.18, $R(0)$ has an eigenvalue which is zero or purely imaginary. In either case for every $t>0$, $\exp[R(0)t]$ has an eigenvalue which has absolute value 1. Now note that for all i,j,

$$\{\exp[R(0)t]\}_{ij} = \tilde{P}_{ij}(0,t)$$

$$= P[N(t)=0, J(t)=j \mid N(0)=0, J(0)=i]$$

$$\leq P[J(t)=j \mid N(0)=0, J(0)=i]$$

$$= \tilde{P}_{ij}(1,t) = [\exp(Q^*t)]_{ij}.$$

Since 1 is the absolute value of an eigenvalue of $\exp[R(0)t]$, by (R1) and (R2) and the sub-stochasticity of $\tilde{P}(0,t)$, we now have that the PF-eigenvalue of $\tilde{P}(0,t)$ is 1. Our assumption of the irreducibility of Q^* and (R3) now imply that

$$\exp[R(0)t] \equiv \exp[Q^*t] \quad \text{for all } t>0,$$

or

$$R(0) = Q^*.$$

Thus we have proven

<u>Theorem 1.3.19</u>: If $R(0) \neq Q^*$, then $R(0)$ is stable.

<u>Remark</u>: The condition $R(0) \neq Q^*$ is equivalent to asserting that for some i, $1 \leq i \leq m$, <u>at least</u> one of the conditions

a) $\lambda_i[1-\phi_i(0)]>0$,

b) for some j, $T_i^\circ \alpha_j[1-\Phi_{ij}(0)]>0$,

or

c) for some $j \neq i$, $T_{ij}[1-\psi_{ij}(0)]>0$,

is true. From the definition of these quantities above, it is clear that if the above condition is not met then the N-Process cannot develop beyond zero.

We shall from now on make the assumption of non-triviality of the N-Process, viz., $R(0) \neq Q^*$, so that the conclusion of Theorem 1.3.19 holds. A useful consequence of this assumption we now record as

Corollary 1.3.20: $[sI-R(0)]^{-1}$ exists for all $s \geqq 0$.

Proof: Since $R(0)$ is stable, every eigenvalue of $sI-R(0)$, for $s \geqq 0$, has positive real part and hence the result.

1.4 The N/G/1 Queue and the imbedded semi-Markov Sequence

We consider a single server queue in which arrivals occur according to an N-Process defined in the previous section, and the service times of successive customers are independent identically distributed random variables. It is assumed that the input and the service processes are mutually independent. Such a model will be denoted by N/G/1.

For the purpose of discussing queue length, busy period etc., it is clear that the order of service is immaterial; all that we shall assume are that the server cannot idle as long as there are customers in the system, and that having started a customer's service, the server must proceed to its conclusion without interruption. In Chapter V, for the purpose of discussing the virtual waiting time alone, we shall make the additional assumption that the server must serve the groups in the order of their arrival, the order of service within each group, once again, being arbitrary.

For describing the N-Process characterizing the input we shall use the same notations used in Section 1.2. The service time c.d.f. assumed to be non-degenerate will be denoted by $\tilde{H}(\cdot)$, its Laplace-Stieltjes transform (LST) by $H(\cdot)$ and its moments (about the origin), whenever they exist, by $\mu^{(i)}$, $i=1,2,\ldots$.

We now define the r.v.s. $\{\tau_n: n \geqq 0\}$ as the successive epochs of departure and assume $\tau_0=0$. Defining X_n and J_n to be respectively the queue length (i.e., the number of customers in the system) and the phase of the N-Process at τ_n+, it is easily seen that $\{(X_n, J_n, \tau_{n+1}-\tau_n): n \geqq 0\}$ form a semi-Markov sequence with state-space $\{0,1,\ldots\}\times\{1,\ldots,m\}$ and transition probability matrix $\tilde{Q}(\cdot)$ given by

$$\tilde{Q}(x) = \begin{bmatrix} \tilde{B}_0(x) & \tilde{B}_1(x) & \tilde{B}_2(x) & \ldots & \ldots \\ \tilde{A}_0(x) & \tilde{A}_1(x) & \tilde{A}_2(x) & \ldots & \ldots \\ 0 & \tilde{A}_0(x) & \tilde{A}_1(x) & \ldots & \ldots \\ 0 & 0 & \tilde{A}_0(x) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \quad x \geqq 0, \tag{1.4.1}$$

where the m×m matrices of mass functions

$$\tilde{A}_n(x) = \int_{0-}^{x} P(n,u)\,d\tilde{H}(u), \quad n \geqq 0,\ x \geqq 0, \tag{1.4.2}$$

$$\tilde{B}_n(x) = \sum_{k=1}^{n+1} (\tilde{U}_k * \tilde{A}_{n-k+1})(x), \quad n \geqq 0,\ x \geqq 0, \tag{1.4.3}$$

and

$$\tilde{U}_k(x) = \{\int_{0-}^{x} P(0,y)\,dy\}\{T^{\circ}A^{\circ}r(k) + Tq(k) + \Delta(\underline{\lambda})\Delta(\underline{p}(k))\},$$
$$k \geqq 1,\ x \geqq 0 \tag{1.4.4}$$

where $r(k)$ and $q(k)$ are m×m matrices with respective entries $r_{ij}(k)$ and $q_{ij}(k)$, $\underline{p}(k)$ is an m-vector with entries $p_i(k)$

and $\Delta(\underline{p}(k))$ is an $m \times m$ diagonal matrix with $\underline{p}(k)$ along the diagonal. Also $*$ in the definition of $\tilde{B}_n(\cdot)$ in (1.4.3) denotes matrix convolution. We note that the (i,j)-th entry of $\tilde{U}_k(x)$ is the conditional probability, given $J(0)=i$, that the first arrival occurs at or before x and is of group size k, and that the phase of the N-Process at the epoch of the first arrival is j.

We now introduce the following notations for use in the sequel. $R^{ij}_{k\ell}(x)$ will denote the renewal function giving the expected number of visits in $[0,x]$ to (k,ℓ) by the Markov Renewal Process defined by $\tilde{Q}(\cdot)$ given that the initial state is (i,j). Also m(i,j) will denote the mean recurrence time of the state (i,j) in the Markov Renewal Process $\tilde{Q}(\cdot)$. For the results governing these quantities we refer the reader to Çinlar [2] and Hunter [8].

Before concluding this section we point out that most of the results in the sequel are obtained by studying the imbedded semi-Markov sequence described above, and we shall invoke many a result from the literature governing semi-Markov and Markov Renewal processes. The basic definitions and results on these processes are by now quite well-known, and an excellent account of these may be found in the work of Çinlar [2]. Among the basic references in this connection [2,8,18,24,25], we draw particular attention to those of Çinlar [2], Hunter [8] and Neuts [18].

1.5 Some Special Cases of the N/G/1 model

Below we present a few interesting special cases of the N/G/1 model. Most of the material below is based on Neuts [22] and presented here for completeness.

(a) PH/G/1 Queues: In the definition of the N-Process if we set $\lambda_1 = \ldots = \lambda_m = 0$, $\psi(z) \equiv E$, $\Phi(z) = p(z)E$, where E is an $m \times m$ matrix each of whose entries is 1, and where $p(z)$ is the p.g.f. of the group size, then we get the PH/G/1 queue (with group arrivals) wherein the inter-arrival times are i.i.d. phase type with c.d.f. $F(\cdot)$ given by (1.2.2). As pointed out earlier, queues with exponential, generalized Erlang and hyper-exponential inter-arrival times are but few of the special cases of this large class whose versatality stems from the closure properties of Phase Type distributions proven in [14].

While some of the very special cases in this class such as the M/G/1 and E_k/G/1 models have been discussed earlier in the literature, there is no systematic discussion of PH/G/1 queues in their generality. The nearest attempts at this are the work of Carson [1] who discussed computational methods for PH/PH/1 queues, i.e., queues where both inter-arrival and service times are of Phase Type and that of Cox [3] who discussed queues with "rational arrival processes", a class of processes which is only slightly more general than PH-Renewal Processes. In this connection we point out that the present theory on rational arrival processes, due to its heavy reliance on complex arithmetic, is not computationally very attractive. As will be seen in the sequel, the formulas in this paper are presented in a form computable in real arithmetic.

(b) Superposition of a Poisson Process and a PH-Renewal Process: Kuczura [10] considers a queue whose input process is the superposition of a Poisson and a renewal process where the inter-arrival times of the latter have a rational Laplace-Stieltjes transform. This is only slightly more general than considering the superposition of a Poisson Process and a PH-Renewal Process. In [22] Neuts has pointed out the practical merit of considering queues of this type where the Poisson Process describes a "background input" and the PH-Renewal Process (with group arrivals) describes "burst inputs". Such a process corresponds to a given matrix T and a vector $\underline{\alpha}$ and the parameter choices $\lambda_i \equiv \lambda$, $\phi_i(z) \equiv z$, $\Phi(z) = p(z)E$, $\psi(z) = E$, where $p(z)$ is the p.g.f. of the group size in the renewal arrival process and E is an $m \times m$ matrix with each entry equal to 1.

Stochastic models which involve superposition of (even as few as two) general renewal processes are, in most cases, intractable. The results in the sequel become important when one notes that the N-Process contains as special cases a large number of such complex models as the one described above.

(c) Queues with Markov-Modulated Poisson Arrivals: If in the definition of the N-Process we set $\psi(z)=\Phi(z)=E$ where E is an m×m matrix of 1's, and $\phi_i(z)=z$ for $1 \leq i \leq m$, then we obtain the Markov-Modulated Poisson arrival process which has been used by several authors [13,21,23,28] to describe the input to queues. Such a process can be used to model a large variety of queueing phenomena such as rush-hour behavior and others. The work of Heffes [7], which, in a telephone engineering context, deals with the interrupted Poisson Process in which arrivals occur on alternating exponentially distributed intervals, is of this type and corresponds to the choice

$$T = \begin{bmatrix} -\sigma_1 & \sigma_1 \\ 0 & -\sigma_2 \end{bmatrix}, \quad \underline{T}^\circ = \begin{bmatrix} 0 \\ \sigma_2 \end{bmatrix}, \quad \underline{\alpha} = (1,0)$$

$$\lambda_1=\lambda, \ \phi_1(z)\equiv z, \ \lambda_2=0, \ \phi_2(z) \text{ arbitrary}, \ \Phi(z)=\psi(z)=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The model of Heffes can be easily generalized by defining an interrupted Poisson Process on an alternating renewal process of phase type. For further discussion on this we refer the reader to Neuts [22].

(d) Queues with arrivals inhibited or stimulated by renewals: In his paper [22] Neuts discusses how tractable qualitative models for arrival streams which exhibit an inhibition or stimulation of arrivals for a certain length of time by certain renewal epochs can be modeled as an N-Process. Such models are of considerable practical interest.

A PH-Distribution is called progressive if it has a representation $(\underline{\alpha}, T)$ in which T is upper-triangular. It is easy to see that a PH-Distribution is progressive iff it is a finite mixture of generalized Erlang distributions. Since every path function of the Markov Process Q is then non-decreasing, we can, by suitable choices of the λ_i-parameters of states close to renewals, model any inhibitory or stimulatory effect of the renewal.

The examples presented above should indicate to the reader the wide gamut of queues that are special cases of the N/G/1 model. The ensuing discussion which presents a unified treatment of these special cases in a computationally tractable form, we hope, enhances the merit of the N-Process as a versatile model for describing input to queues.

CHAPTER II

THE BUSY PERIOD AND THE BUSY CYCLE

2.1 Introduction

In this Chapter we show that the Semi-Markov Process $\tilde{Q}(\cdot)$ defined in Section 1.4 is a special case of a general class of such processes studied by Neuts [18]. Appealing to the results proven in [18] we obtain, in terms of an appropriately defined traffic intensity ρ, a necessary and sufficient condition for the stability of the N/G/1 queue and derive the busy period characteristics in terms of the minimal solution in the class of sub-stochastic matrices of a certain non-linear matrix functional equation. The necessary notations and preliminaries to this end are set forth in Section 2.2. In Section 2.3 we discuss the busy period of the N/G/1 queue obtaining the joint transform of the number of services during a busy period and the duration of the busy period. We also obtain the expected number served during a busy period in an easily computable form. The last section provides a similar discussion of the busy cycle.

2.2 Notations and Preliminaries

Throughout this paper we shall adopt the convention to denote probability mass functions by upper-case Roman letters superscripted by a tilde and their Laplace-Stieltjes transforms (LST's) by the same letters without the tilde. Thus $U_k(\cdot)$, $A_n(\cdot)$ and $B_n(\cdot)$ will respectively denote the LST's of $\tilde{U}_k(\cdot)$, $\tilde{A}_n(\cdot)$ and $\tilde{B}_n(\cdot)$. The values of these LST's at 0+ are respectively denoted by U_k, A_n and B_n. We let

$$\tilde{A}(x) = \sum_{n=0}^{\infty} \tilde{A}_n(x) \tag{2.2.1}$$

and denote the LST of $\tilde{A}(\cdot)$ by $A(\cdot)$. Also $A=A(0+)$. We now define the generating functions

$$A(z,s)=\sum_{n=0}^{\infty} A_n(s)z^n, \quad |z|\leqq 1,\ \text{Re } s\geqq 0, \tag{2.2.2}$$

and

$$U(z,s)=\sum_{k=1}^{\infty} U_k(s)z^k, \quad |z|\leqq 1,\ \text{Re } s\geqq 0, \tag{2.2.3}$$

and note that $A=A(1,0)$, $U=U(1,0)$.

Lemma 2.2.4

For $|z|\leqq 1$, Re $s\geqq 0$,

$$U(z,s)=[sI-R(0)]^{-1}[R(z)-R(0)] \tag{2.2.5}$$

Proof

$$U(z,s)=\sum_{k=1}^{\infty} z^k\int_{0-}^{\infty} e^{-sx}\, d\tilde{U}_k(x)$$

$$=\left[\int_{0-}^{\infty} e^{-sx}P(0,x)dx\right]\left[\sum_{k=1}^{\infty} z^k\{T^\circ A^\circ\text{or}(k)+T\text{o}q(k)+\Delta(\underline{\lambda})\Delta(\underline{p}(k))\}\right]$$

by (1.4.4)

$$=[sI-R(0)]^{-1}[R(z)-R(0)]$$

by Corollary 1.3.20, and the fact that

$$\sum_{k=1}^{\infty} z^k\{T^\circ A^\circ \circ r(k)+T\circ q(k)+\Delta(\underline{\lambda})\Delta(\underline{p}(k))\}=R(z)-R(0)$$

which is an easy consequence of (1.3.4).

Basic to our discussion of the busy period and the busy cycle are the first passage times of the semi-Markov process $\tilde{Q}(\cdot)$ from the set of states $\underline{i+1}=\{(i+1,j): 1\leqq j\leqq m\}$ to the set of states $\underline{i}=\{(i,j): 1\leqq j\leqq m\}$. We now set up a number of notations to describe these first passages.

Let $\tilde{G}_{jj'}^{[i]}(k,x)$ be the probability that, given that the semi-Markov process $\tilde{Q}(\cdot)$ starts in the state (i,j), it reaches the set $\underline{0}$ for the first time after k transitions by visiting the state $(0,j')$ and the time of such a first passage is atmost x. The matrix $\tilde{G}^{[i]}(k,x)$ will have the entries $\tilde{G}_{jj'}^{[i]}(k,x)$, $1\leqq j,j'\leqq m$.

In particular, the matrix $\tilde{G}^{[1]}(k,x)$ will be denoted by $\tilde{G}(k,x)$. The sequence of matrices $\{\tilde{G}(k,x): k\geqq 0\}$, $x\geqq 0$, defines completely the first passage time distributions from $\underline{1}$ to $\underline{0}$, and, as noted by Neuts [18], in view of the structure of $\tilde{Q}(\cdot)$ also from $\underline{i+1}$ to $\underline{i}$, for $i\geqq 0$. We define the transform

$$G(z,s)=\sum_{k=1}^{\infty}\int_{0-}^{\infty} e^{-sx}d\tilde{G}(k,x)z^k, \tag{2.2.6}$$

for $|z|\leqq 1$ and Re $s\geqq 0$. For notational convenience we shall write $G(1-,0+)$ as G.

Noting that $\tilde{Q}(\cdot)$ has a structure same as the general class of such matrices discussed by Neuts [18], we can specialize the general results to the case at hand. Before we state the relevant results, following Neuts [18], let us give the

Definition 2.2.7: The semi-Markov process $\tilde{Q}(\cdot)$ is boundary leading iff $G>>0$.

The following theorem establishes the boundary leading property of the semi-Markov process $\tilde{Q}(\cdot)$.

Theorem 2.2.8:

(i) A is irreducible and stochastic.

(ii) The diagonal entries of A_0 are all positive.

Proof:

(i) follows from the irreducibility of Q^* and the non-degeneracy of $\tilde{H}(\cdot)$, in view of the relation

$$A = \int_{0-}^{\infty} \exp(Q^*t)d\tilde{H}(t). \tag{2.2.9}$$

(ii) is obvious by noting that for every $1 \leqq i \leqq m$,

$$A_0(i,i) = \int_{0-}^{\infty} P_{ii}(0,x)d\tilde{H}(x)$$

and that $P_{ii}(0,x)>0$ for every $x \geqq 0$.

Corollary 2.2.10: The semi-Markov process $\tilde{Q}(\cdot)$ is boundary leading.

Proof: By Theorem 2.2.8 and the structure of $\tilde{Q}(\cdot)$, it is seen that $\tilde{Q}(\infty)$ is irreducible. It is obvious that this implies the boundary leading property.

We now state the basic results obtained by Neuts [18] as

Theorem 2.2.11:

(i) If we define $G^{[i]}(z,s)$ as the analogous transform of $\{\tilde{G}^{[i]}(k,x): k \geqq 0,\ x \geqq 0\}$, then $G^{[i]}(z,s)$ is the i-th power of the matrix $G(z,s)$.

(ii) $G(z,s)$ satisfies the non-linear matrix functional equation

$$G(z,s)=z \sum_{n=0}^{\infty} A_n(s)G^n(z,s)=zA(G(z,s),s) \qquad (2.2.12)$$

where $A(z,s)$ is as in (2.2.2).

(iii) For $0 \leqq z \leqq 1$, $s>0$, there exists a unique nonnegative matrix $G(z,s)$ which satisfies Equation (2.2.12). The entries of $G(z,s)$ are analytic functions of z and s, and the matrix $G(z,s)$ may be written in the form (2.2.6), and the entries of all matrices $\tilde{G}(k,\cdot)$, $k \geqq 0$ are probability mass functions. The matrices $\tilde{G}(k,\infty)$, $k \geqq 0$ are all nonnegative, and the matrix

$$G = G(1-,0+) = \sum_{k=1}^{\infty} \tilde{G}(k,\infty), \qquad (2.2.13)$$

defined by continuity, is sub-stochastic.

(iv) Let $\rho=\underline{\pi}\underline{\beta}$, where $\underline{\pi}$ is the invariant probability vector of A and

$$\underline{\beta} = \sum_{n=1}^{\infty} nA_n\underline{e}. \qquad (2.2.14)$$

If $\rho \leqq 1$, the matrix G is stochastic. If $\rho>1$, at least one component of $G\underline{e}$ is less than one.

(v) The matrix G is the minimal solution, in the class of sub-stochastic matrices, of (2.2.12) with $z=1$, $s=0$, and can be computed by the recurrence relations

$$\left.\begin{aligned} G_0 &= 0 \\ G_{n+1} &= \sum_{\nu=0}^{\infty} A_\nu G_n^\nu, \quad n \geq 0 \end{aligned}\right\} . \tag{2.2.15}$$

The matrices $\{G_n : n \geq 0\}$ defined above are non-decreasing.

(vi) The Markov Renewal Process defined by $\tilde{Q}(\cdot)$ is positive recurrent, null recurrent or transient according as ρ is less than, equal to or greater than 1.

Remarks:

(i) The equation (2.2.12) is the analogue of Takács' equation [26] for the M/G/1 queue.

(ii) The recurrence relation (2.2.15) yields rapid convergence and thus facilitates easy computation of the matrix G.

Theorem 2.2.16:

(i) $\underline{\pi}=\underline{\theta}$, where $\underline{\theta}$ is the invariant probability vector of Q^*.

(ii) $\rho=\xi^*\mu^{(1)}$, where ξ^* is the "arrival rate" given by (1.3.14) and $\mu^{(1)}$ is the mean service time.

(iii) The N/G/1 queue is stable iff $\rho<1$.

Proof:

(i) follows from (2.2.9) and the uniqueness of the invariant probability vector.

(ii) Noting that

$$\underline{\beta} = \sum_{n=1}^{\infty} nA_n \underline{e} = \frac{\partial}{\partial z} A(z,s)\underline{e}\Big|_{\substack{z=1- \\ s=0+}}$$

$$= \int_{0-}^{\infty} M(t)\underline{e}\, d\tilde{H}(t),$$

we have by (1.3.8)

$$\underline{\beta} = \int_{0-}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\nu=0}^{n-1} Q^{*\nu} R'(1)\, Q^{*n-1-\nu}\, \underline{e} \right\} d\tilde{H}(t),$$

whence,

$$\rho = \underline{\theta}\underline{\beta} = \xi^* \mu^{(1)}$$

in view of (1.3.14) and the fact that $\underline{\theta}Q^* = \underline{0}$.

(iii) is only a re-statement of Theorem 2.2.11 (vi).

Remark: In view of the remark following Theorem 1.3.15, ρ may be called the traffic intensity of the N/G/1 queue. The N/G/1 queue is stable iff the traffic intensity is less than 1. In the sequel we shall always assume that $\rho<1$.

2.3 The Busy Period

Note that from its definition it is clear that $G^{[k]}(z,s)$ for $k\geq 1$ completely specifies the busy period which starts with k customers of the N/G/1 queue. In this section we will be concerned with the first busy period of the N/G/1 queue given that the queue starts with no customers at time 0. As will be evident later, this discussion will be found useful in determining the invariant probability vector of $\tilde{Q}(\infty)$.

In the last section we noted that the matrix G is strictly positive and stochastic. We let $\underline{g}$ denote the invariant probability vector of G and define $G^{\circ\circ}$ to be the m×m matrix each of whose rows is $\underline{g}$. Defining μ_j to be the expected first passage time from (i+1,j) to $\underline{i}$, $i \geq 0$, in the semi-Markov process $\tilde{Q}(\cdot)$ and $\tilde{\mu}_j$ to be the expected number of service completions during such a first passage, we can easily prove the following result which yields the vectors $\underline{\mu}$ and $\underline{\tilde{\mu}}$ whose j-th components are respectively μ_j and $\tilde{\mu}_j$.

<u>Theorem 2.3.1</u>:

$$\underline{\tilde{\mu}} = (I-G+G^{\circ\circ})[I-A+G^{\circ\circ}-\Delta(\underline{\beta})G^{\circ\circ}]^{-1}\underline{e} \tag{2.3.2}$$

and

$$\underline{\mu}=(I-G+G^{\circ\circ})[I-A+G^{\circ\circ}-\Delta(\underline{\beta})G^{\circ\circ}]^{-1}\mu^{(1)}\underline{e}, \tag{2.3.3}$$

where $\Delta(\underline{\beta})=\mathrm{diag}(\beta_1,\dots,\beta_m)$ and $\underline{\beta}$ is as in (2.2.14).

<u>Proof</u>: It is shown in [18] that

$$\underline{\tilde{\mu}}=(I-G+G^{\circ\circ})[I-A+G^{\circ\circ}-\Delta(\underline{\beta})G^{\circ\circ}]^{-1}\underline{e}$$

and

$$\underline{\mu}=(I-G+G^{\circ\circ})[I-A+G^{\circ\circ}-\Delta(\underline{\beta})G^{\circ\circ}]^{-1}\sum_{n=0}^{\infty} A_n^{(1)}\underline{e},$$

where

$$A_n^{(1)} = \int_{0-}^{\infty} x\, d\tilde{A}_n(x).$$

The theorem follows by noting that

$$\sum_{n=0}^{\infty} A_n^{(1)}\underline{e} = \int_{0-}^{\infty} x\, d\tilde{H}(x)\underline{e} = \mu^{(1)}\underline{e}.$$

Remarks: Higher moments of the duration of the first passage times and the number served during such first passages can be found by differentiation of equation (2.2.12). We refer the reader to Neuts [18] for the formulas and computational methods governing these quantities.

Corollary 2.3.4:

$$\underline{\mu} = \mu^{(1)} \; \underline{\tilde{\mu}} \tag{2.3.5}$$

Remark: Equation (2.3.5) simply states that the expected first passage time is the product of the expected number served during such first passage and the expected duration of each service, a result which is intuitively quite obvious.

Corollary 2.3.6:

$$\underline{g} \; \underline{\tilde{\mu}} = (1-\rho)^{-1} \tag{2.3.7}$$

$$\underline{g} \; \underline{\mu} = \mu^{(1)} \; (1-\rho)^{-1} \tag{2.3.8}$$

Proof: These follow easily from (2.3.2) and (2.3.3).

Remark: The formulas (2.3.5), (2.3.7) and (2.3.8) provide powerful computational checks on the accuracy of numerical computations of $\underline{\tilde{\mu}}$ and $\underline{\mu}$ using Theorem 2.3.1. Having computed G using (2.2.15), with a little additional effort one can easily compute $\underline{g}$ using an algorithm such as Wachter's method [27].

We now define L(z,s) to be the joint transform of the number served and the duration of the first busy period of the

N/G/1 queue, given that at time 0 there are no customers in the system. The matrix of mass functions $\tilde{L}(k,x)$ associated with $L(z,s)$ is such that its (j,j')-th entry is the conditional probability, given $X(0)=0$ and $J(0)=j$, that the first busy period of the N/G/1 queue is of duration less than or equal to x and consists of k services and that at the epoch where the busy period ends the phase of the N-Process is j'. A direct probabilistic argument yields

<u>Theorem 2.3.9</u>:

$$L(z,s)=\sum_{k=1}^{\infty} U_k(0)\, G^k(z,s)=U(G(z,s),0) \tag{2.3.10}$$

where $U(z,s)$ is given by (2.2.3).

Defining $\tilde{\mu}_j^*$ and μ_j^* to be respectively the mean number served during and the mean duration of the first busy period given $X(0)=0$, $J(0)=j$, and $\underline{\tilde{\mu}}^*$ and $\underline{\mu}^*$ to be the vectors with $\tilde{\mu}_j^*$ and μ_j^* as their respective j-th entries, $1 \leq j \leq m$, we can now prove

<u>Theorem 2.3.11</u>:

$$\underline{\tilde{\mu}}^*=[U(1,0)-U(G,0)-R^{-1}(0)R'(1)G^{\circ\circ}](I-G+G^{\circ\circ})^{-1}\underline{\tilde{\mu}} \tag{2.3.12}$$

$$\underline{\mu}^*=[U(1,0)-U(G,0)-R^{-1}(0)R'(1)G^{\circ\circ}](I-G+G^{\circ\circ})^{-1}\underline{\mu} \tag{2.3.13}$$

<u>Proof</u>:

$$\underline{\tilde{\mu}}^* = \frac{\partial}{\partial z} L(z,s)\underline{e}\,\Big|_{\substack{z=1- \\ s=0+}}$$

$$= \sum_{k=1}^{\infty} U_k(0) \sum_{\nu=0}^{k-1} G^{\nu}\underline{\tilde{\mu}} \qquad \text{by (2.3.10)}$$

$$=\left[\sum_{k=1}^{\infty} U_k(0)(I-G^k+kG^{\circ\circ})\right](I-G+G^{\circ\circ})^{-1}\tilde{\underline{\mu}}$$

$$=[U(1,0)-U(G,0)-R^{-1}(0)R'(1)G^{\circ\circ}](I-G+G^{\circ\circ})^{-1}\tilde{\underline{\mu}}.$$

The third equality above is obtained by rewriting

$$\sum_{\nu=0}^{k-1} G^{\nu}=\left\{\sum_{\nu=0}^{k-1} G^{\nu}(I-G+G^{\circ\circ})\right\}(I-G+G^{\circ\circ})^{-1}$$

and using the fact that $GG^{\circ\circ}=G^{\circ\circ}G=G^{\circ\circ}$. The last equality follows from

$$\sum_{k=1}^{\infty} k\, U_k(0) = -R^{-1}(0)R'(1),$$

a formula provable by differentiating (2.2.5) with respect to z and setting $z=1-$, $s=0+$.

The proof of (2.3.13) is analogous and hence omitted.

Corollary 2.3.14:

$$\underline{\mu}^* = \mu^{(1)}\ \tilde{\underline{\mu}}^* \qquad (2.3.15)$$

Remarks: Although the formulas (2.3.12) and (2.3.13) above appear to be rather complicated, in actual practice they are well-suited to numerical computations. Note that $U(1,0)=I-R^{-1}(0)(T+T^{\circ}A^{\circ})$ while implementing these formulas. The formula (2.3.15) is, once again, quite intuitive.

2.4 The Busy Cycle

The subject matter of this section are the successive returns of the semi-Markov process $\tilde{Q}(\cdot)$ to the level $\underline{0}$. Let $K_0(z,s)$ be the joint transform of the number served during and the duration of the busy cycle. $\tilde{K}_0(n,x)$, the matrix of mass functions, of which $K(z,s)$ is the transform is such that its (j,j')-th entry yields the conditional probability, given that the busy cycle starts in phase j, that the busy cycle consists of n services, is of duration atmost x and ends in phase j'. By a direct probabilistic argument we obtain

<u>Theorem 2.4.1</u>:

$$K_0(z,s)=[sI-R(0)]^{-1}[-R(0)]L(z,s) \tag{2.4.2}$$

where $L(z,s)$ is given by (2.3.10).

<u>Proof</u>:

$$\begin{aligned}
K_0(z,s) &= \sum_{\nu=0}^{\infty} zB_\nu(s)G^\nu(z,s)\\
&= \sum_{\nu=0}^{\infty} z\left\{\sum_{k=1}^{\nu+1} U_k(s)A_{\nu-k+1}(s)\right\} G^\nu(z,s)\\
&= \sum_{k=1}^{\infty} U_k(s)G^k(z,s), \text{ using (2.2.12)}\\
&=U(G(z,s),s)\\
&=[sI-R(0)]^{-1} \sum_{k=1}^{\infty} \{T°A°or(k)+Toq(k)+\Delta(\underline{\lambda})\Delta(\underline{p}(k))\}G^k(z,s)\\
&=[sI-R(0)]^{-1}[-R(0)] \sum_{k=1}^{\infty} U_k(0)G^k(z,s)\\
&=[sI-R(0)]^{-1}[-R(0)]L(z,s).
\end{aligned}$$

By evaluating $-\frac{\partial}{\partial s} K_0(1,s)\underline{e}\Big|_{s=0+}$, we can easily prove

Theorem 2.4.2: Let $\tilde{\tilde{\mu}}_j$ be the mean duration of a busy cycle starting in phase j, $1 \leq j \leq m$. The vector $\underline{\tilde{\tilde{\mu}}}$ whose j-th entry is $\tilde{\tilde{\mu}}_j$ is given by

$$\underline{\tilde{\tilde{\mu}}} = \underline{\mu}^* - R^{-1}(0)\underline{e} \qquad (2.4.3)$$

Remark: Note that the j-th entry of $-R^{-1}(0)\underline{e} = \int_{0-}^{\infty} P(0,y)\underline{e}\, dy$ is the expected duration of an idle period starting in phase j. In view of this, (2.4.3) is simply the statement that the expected duration of the busy cycle is the expected duration of the idle period plus the expected duration of the busy period following the idle period, a result which again is intuitively obvious!

We conclude this Chapter by pointing out that it appears possible to simplify many of the formulas above in certain special cases. While we shall not pursue this line in detail, we present below a few results for the PH/G/1 queue to illustrate our point.

Special Case: PH/G/1 queue (with single arrivals)

In this case since $R(0)=T$, $D=E$, $C=0$, $\underline{\gamma}=\underline{0}$ and $U(z,0)=A^{\circ\circ}z$, many simplifications occur. For example, we have

$$\tilde{\underline{\mu}}^* = (\underline{\alpha}\tilde{\underline{\mu}})\underline{e} \qquad (2.4.4)$$

as is seen by specializing in (2.3.11).

Further, in this case it may be verified that

$$K_0(z,s)=(sI-T)^{-1}T^\circ A^\circ G(z,s)$$

$$=\{(sI-T)^{-1}\underline{T}^\circ\}\{\underline{\alpha}G(z,s)\}$$

showing that the idle period and the busy period are independent.

CHAPTER III

THE STATIONARY QUEUE LENGTH DISTRIBUTIONS

3.1 Introduction

In this Chapter we discuss the stationary distributions of the queue length (i.e., the number of customers in the system) at a point of departure and at an arbitrary epoch t. In general, these two are shown to be different. Section 3.2 discusses the stationary distribution of the queue length at a point of departure. This is followed by a discussion of the stationary queue length distribution at an arbitrary epoch t in Section 3.3. We show that the stationary probability that the server is idle at an arbitrary epoch t is $(1-\rho)$ where ρ is the traffic intensity defined earlier - a result which is pleasantly surprising and not too obvious in view of the non-recurrent nature of the input.

3.2 Queue length at epochs of departure

The stationary queue length density at the point of departure is denoted by $\underline{x}$ and is obtained by computing the invariant probability vector of the irreducible stochastic matrix $\tilde{Q}(\infty)$ which under the assumption $\rho<1$ is ergodic. The defining system of equations

$$\underline{x}\tilde{Q}(\infty)=\underline{x}, \quad \underline{x}\underline{e}=1 \tag{3.2.1}$$

can, after partitioning the vector $\underline{x}$ as

$$\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots.), \tag{3.2.2}$$

be written as

$$\underline{x}_i = \underline{x}_0 B_i + \sum_{k=1}^{i+1} \underline{x}_k A_{i-k+1}, \quad i \geqq 0. \tag{3.2.3}$$

Multiplying in (3.2.3) by z^i and summing over $i \geqq 0$, we can easily obtain

<u>Theorem 3.2.4</u>: The generating function

$$\underline{X}(z) = \sum_{i=0}^{\infty} \underline{x}_i z^i, \quad |z| \leqq 1, \tag{3.2.5}$$

satisfies the equation

$$\underline{X}(z)[zI - A(z,0)] = \underline{x}_0[U(z,0) - I]A(z,0) \tag{3.2.6}$$

where $U(z,s)$ and $A(z,s)$ are given by (2.2.3) and (2.2.2) respectively.

<u>Corollary 3.2.7</u>:

$$\underline{X}(1-) = \sum_{i=0}^{\infty} \underline{x}_i = -\underline{x}_0 R^{-1}(0)(T + T°A°)A(I - A + \Theta)^{-1} + \underline{\theta} \tag{3.2.8}$$

<u>Proof</u>: Let $z \to 1-$ in (3.2.6). We get

$$\underline{X}(1-)[I-A] = \underline{x}_0[U(1,0) - I]A.$$

Adding $\underline{X}(1-)\Theta$ to both sides and noting that $\underline{X}(1-)\Theta = (\underline{X}(1-)\underline{e})\underline{\theta} = \underline{\theta}$ and $U(1,0) - I = -R^{-1}(0)(T + T°A°)$, we get

$$\underline{X}(1-)(I - A + \Theta) = -\underline{x}_0 R^{-1}(0)(T + T°A°)A + \underline{\theta}.$$

(3.2.8) follows from the non-singularity of $(I - A + \Theta)$ and the fact that $\underline{\theta}(I - A + \Theta) = \underline{\theta}$.

Corollary 3.2.9: $\underline{X}(1-)\underline{e}=1$

Remarks: In practice, the system of equations (3.2.3) is solved by truncating the number of equations at a sufficiently large value of the index i and then applying an algorithm such as Gauss-Seidel. In the next chapter we shall derive the first two moments of the queue length with which one may truncate the system (3.2.3) using a "$\mu+3\sigma$ limit". Below we will provide a method of computing $\underline{x}_0$ directly using which $\underline{X}(1-)$ may be computed through (3.2.8). Equation (3.2.8) provides an excellent computational check on the numerical computation of $\underline{x}$.

Below we give an alternate method for determining the vectors $\underline{x}_0$ and $\underline{x}_1$ which can be effectively used in the numerical computation of $\underline{x}$ for initialization purposes in an algorithm such as Gauss-Seidel.

Lemma 3.2.10: The matrix L(1,0) defined by continuity in (2.3.10) is irreducible and stochastic.

Proof: We have

$$L(1,0)=U(G,0)=-R^{-1}(0)\sum_{k=1}^{\infty}\{T°A°\text{or}(k)+T\text{oq}(k)+\Delta(\underline{\lambda})\Delta(\underline{p}(k))\}G^k$$

from which the stochasticity of L(1,0) is easily verified. Now, since the strictly positive stochastic matrices $G^k \to G^{\circ\circ}$ as $k\to\infty$, there exists an $\epsilon>0$ such that

$$G^k \geqq \epsilon E \quad \text{for all } k\geqq 1,$$

where E is the $m \times m$ matrix with each entry equal to one.

Then

$$L(1,0) \geqq -R^{-1}(0)[R(1)-R(0)]\epsilon E = \epsilon E >> 0,$$

for

$$\sum_{k=1}^{\infty} \{T^\circ A^\circ \circ r(k) + T \circ q(k) + \Delta(\underline{\lambda})\Delta(\underline{p}(k))\} = R(1)-R(0)$$

and

$$R(1)\underline{e} = (T+T^\circ A^\circ)\underline{e} = \underline{0}.$$

In the sequel we let $\underline{\kappa}_0$ denote the invariant probability vector of $L(1,0)$. The computation of $\underline{\kappa}_0$, can, once again, be easily done by Wachter's method [27]. We now prove

Theorem 3.2.11:

$$\underline{x}_0 = (\underline{\kappa}_0 \underline{\tilde{\mu}}^*)^{-1} \underline{\kappa}_0 \qquad (3.2.12)$$

where $\underline{\kappa}_0$ is the invariant probability vector of $L(1,0)$ and $\underline{\tilde{\mu}}^*$ is given by (2.3.12).

Proof: The probability $x(0,j)$ is the inverse of the mean recurrence time of the state $(0,j)$ in the Markov chain $\tilde{Q}(\infty)$. That mean recurrence time is clearly the same as the mean recurrence time of $(0,j)$ in the Markov Renewal Process of lattice type $K_0(z,0)=L(z,0)$. By applying Theorem 2.11, p. 196, Hunter [8], the mean recurrence time of $(0,j)$ is given by $(\underline{\kappa}_0 \underline{\tilde{\mu}}^*)/(\underline{\kappa}_0)_j$ where $(\kappa_0)_j$ is the j-th component of $\underline{\kappa}_0$ whence the result.

We now discuss some special cases and show that the above formula for $\underline{x}_0$ particularizes correctly in those situations.

(a) <u>M/G/1 queue</u>: It is trivial here to verify that $x_0=(1-\rho)$, for, $\kappa_0=1$ and $x_0=(\tilde{\mu}^*)^{-1}=(1-\rho)$ in this case.

(b) <u>M/G/1 queue with group arrivals</u>: In this case the above formula for $\underline{x}_0$ simplifies to

$$x_0 = (1-\rho)/\eta,$$

where η is the mean group size, and $\rho=\lambda\eta\mu^{(1)}$.

(c) <u>PH/G/1 queue (with single arrivals)</u>: We have already shown that in this case $L(1,0)=A^{\circ\circ}G$. Now,

$$(\underline{\alpha}G)A^{\circ\circ}G=(\underline{\alpha}G\underline{e})(\underline{\alpha}G)=\underline{\alpha}G$$

whence $\underline{\kappa}_0=\underline{\alpha}G.$

We noted earlier at the end of Chapter II that $\underline{\tilde{\mu}}^*=(\underline{\alpha}\underline{\tilde{\mu}})\underline{e}$. Putting all this in (3.2.11) we get

$$\underline{x}_0 = \frac{1}{\underline{\alpha}\underline{\tilde{\mu}}}(\underline{\alpha}G). \tag{3.2.13}$$

Note the highly intuitive formula

$$\underline{x}_0\underline{e}=1/(\underline{\alpha}\underline{\tilde{\mu}})=1/(\text{mean number served in a busy period})$$

which holds in this case!

To obtain the vector $\underline{x}_1$ we consider the first passage times from the set $\underline{1}=\{(1,j):\ 1\leq j\leq m\}$ to itself. Let $\tilde{K}_1(n,x)$ be an $m\times m$ matrix such that its (i,j)-th entry is the probability that starting in $(1,i)$ the Markov Renewal Process $\tilde{Q}(\cdot)$ returns for the first time to the set $\underline{1}$ in exactly n steps at or before time x and that the phase at the epoch of such a first return is j. Let

$$K_1(z,s)=\sum_{n=0}^{\infty} z^n \int_{0-}^{\infty} e^{-sx}d\tilde{K}_1(n,x), \quad |z|\leq 1, \text{ Re } s\geq 0.$$

Theorem 3.2.14

$$K_1(z,s)=z^2A_0(s)[I-zB_0(s)]^{-1} \sum_{\nu=1}^{\infty} B_\nu(s)G^{\nu-1}(z,s)$$

$$+z\sum_{\nu=1}^{\infty} A_\nu(s)G^{\nu-1}(z,s). \tag{3.2.15}$$

Proof: By a simple probabilistic argument considering the paths which pass through $\underline{0}$ and those that do not, we have

$$K_1(z,s)=zA_0(s)\sum_{r=0}^{\infty} z^rB_0^r(s)\sum_{\nu=1}^{\infty} zB_\nu(s)G^{\nu-1}(z,s)$$

$$+\sum_{\nu=1}^{\infty} zA_\nu(s)G^{\nu-1}(z,s),$$

and simplifying we get (3.2.15).

Corollary 3.2.16: $K_1(1,0)$ is irreducible and stochastic.

Theorem 3.2.17: Let $\underline{\kappa}_1$ denote the invariant probability vector of $K_1(1,0)$. Then

$$\underline{x}_1 = (\underline{\kappa}_1\underline{\kappa}_1^*)^{-1}\underline{\kappa}_1, \tag{3.2.18}$$

where $\underline{\kappa}_1^*=\frac{\partial}{\partial z} K_1(z,0)\Big|_{z=1-}$ is the vector of the mean number of steps in a first passage from $\underline{1}$ to itself and is given by

$$\underline{\kappa}_1^*=\underline{e}+A_0(I-B_0)^{-1}\underline{e} +$$

$$\Bigg[A_0(I-B_0)^{-1}\{\sum_{\nu=1}^{\infty} B_\nu-\sum_{\nu=1}^{\infty} B_\nu G^{\nu-1}+\sum_{\nu=2}^{\infty}(\nu-1)B_\nu G^{\circ\circ}\} +$$

$$(A-A_0)-\sum_{\nu=1}^{\infty} A_\nu G^{\nu-1}+\sum_{\nu=2}^{\infty}(\nu-1)A_\nu G^{\circ\circ}\Bigg](I-G+G^{\circ\circ})^{-1}\underline{\tilde{\mu}} \tag{3.2.19}$$

Proof: Note that $\underline{\kappa}_1^*$ is the vector of mean sojourn times in the Markov Renewal Process $K_1(z,0)$ which is of lattice type. The formula for $\underline{x}_1$ is got by an analogous argument as in Theorem 3.2.11 using a theorem in Hunter [8]. The formula for $\underline{\kappa}_1^*$ given above is obtained by directly computing $\frac{\partial}{\partial z} K_1(z,0)$ using (3.2.15) and simplifying the resulting expression.

Remarks:

(i) It can be verified that the expressions given for $\underline{x}_0$ and $\underline{x}_1$ in (3.2.12) and (3.2.18) respectively do indeed satisfy the steady state equation

$$\underline{x}_0 = \underline{x}_0 B_0 + \underline{x}_1 A_0. \tag{3.2.20}$$

We shall omit the tedious details and refer the reader to Lucantoni [11] for such a verification in a more general set-up.

(ii) Equations (3.2.12) and (3.2.18) are easy to implement and have been successfully used by Lucantoni [11] for numerical computation of $\underline{x}_0$ and $\underline{x}_1$ in more general models than the one discussed here.

(iii) The steady-state equation (3.2.20) provides a powerful computational check for the numerical computation of $\underline{x}_0$ and $\underline{x}_1$ using (3.2.12) and (3.2.18) respectively.

3.3 Queue Length in Continuous Time

In this section we discuss the stationary distribution of the queue length at an arbitrary epoch. We define

$$y(i,j)=\lim_{t\to\infty} P[X(t)=i, J(t)=j \mid X(0)=i', J(0)=j']$$

where $X(t)$ and $J(t)$ denote respectively the queue length and the phase of the N-Process at $t+$. Let $\underline{y}_i$ be the m-vector whose components are $y(i,j)$, $1 \leq j \leq m$ and let $\underline{y}=(\underline{y}_0, \underline{y}_1, \ldots.)$. We also define the generating function

$$\underline{Y}(z) = \sum_{i=0}^{\infty} \underline{y}_i z^i$$

Lemma 3.3.1:

$$\mu^{(1)} - \underline{x}_0 R^{-1}(0)\underline{e} = \mu^{(1)}/\rho = (\xi^*)^{-1} \qquad (3.3.2)$$

Proof: From the relation

$$A(z,0) = \int_{0-}^{\infty} \exp\{R(z)t\} d\tilde{H}(t)$$

we have

$$R(z)A(z,0) = A(z,0)R(z). \qquad (3.3.3)$$

Differentiating this with respect to z, letting $z \to 1-$ and multiplying the resulting equation by $\underline{e}$, we get

$$(T+T^\circ A^\circ)\underline{\beta} = (A-I)R'(1)\underline{e} = (A-I-\Theta)R'(1)\underline{e} + \Theta R'(1)\underline{e} \qquad (3.3.4)$$

Now, differentiating (3.2.6) with respect to z we get on letting $z \to 1-$,

$$\underline{X}'(1-)[I-A] + \underline{X}(1-)[I-A'(1-,0)] =$$
$$\underline{x}_0[U(1,0)-I]A'(1-,0) + \underline{x}_0 U'(1-,0)A$$

Adding $\underline{X}'(1-)\Theta$ to both sides and multiplying by $\underline{e}$ we get

$$\underline{X}'(1-)\underline{e} + 1 - \underline{X}(1-)\underline{\beta} = -\underline{x}_0 R^{-1}(0)(T+T^\circ A^\circ)\underline{\beta} - \underline{x}_0 R^{-1}(0)R'(1)\underline{e} + \underline{X}'(1-)\underline{e}$$

In the above if we substitute the value of $\underline{X}(1-)$ using (3.2.8) and the value of $(T+T°A°)\underline{\beta}$ using (3.3.4) and simplify, we get

$$-\underline{x}_0 R^{-1}(0)\theta R'(1)\underline{e}=(1-\rho) \qquad (3.3.5)$$

or

$$(-\underline{x}_0 R^{-1}(0)\underline{e})(\underline{\theta} R'(1)\underline{e})=(1-\rho).$$

That is $-\underline{x}_0 R^{-1}(0)\underline{e}=(1-\rho)/\xi^*$. The Lemma follows by noting that $\xi^*=\underline{\theta} R'(1)\underline{e}=\rho/\mu^{(1)}$.

<u>Theorem 3.3.6</u>:

$$\underline{y}_0 = -\xi^* \underline{x}_0 R^{-1}(0) \qquad (3.3.7)$$

<u>Proof</u>: Recalling that $R_{ij}^{i'j'}(\cdot)$ are the renewal functions of the Markov Renewal Process $\tilde{Q}(\cdot)$, we can write by a standard argument considering the state of the semi-Markov process $\tilde{Q}(\cdot)$ at the epoch of the last transition before t,

$$P\{X(t)=0,J(t)=j|X(0)=i',J(0)=j'\}=\sum_{k=1}^{m}\int_{0-}^{t} dR_{0k}^{i'j'}(u)P_{kj}(0,t-u).$$

Letting $t\to\infty$ in the above equation and applying the Key Renewal Theorem (Theorem 6.3, p. 153, Çinlar [2]), we get

$$y(0,j)=\sum_{k=1}^{m} m^{-1}(0,k)\int_{0-}^{\infty} P_{kj}(0,u)du \qquad (3.3.8)$$

where $m(0,k)$ is the mean recurrence time of $(0,k)$ in $\tilde{Q}(\cdot)$. By considering the Markov Renewal Process $K_0(1,s)$ it is easily seen that $m(0,k)$ is also the mean recurrence time of $(0,k)$ in this Markov Renewal Process. By Theorem 2.11, p. 196, Hunter [8], we have

$$m(0,j)=(\underline{\kappa}_0\tilde{\tilde{\mu}})/(\underline{\kappa}_0)_j, \quad 1\leq j\leq m \tag{3.3.9}$$

where $\underline{\kappa}_0$ is the invariant probability vector of $K_0(1,0)=L(1,0)$ and $\tilde{\tilde{\mu}}$ is the vector of mean durations of the busy cycle given by (2.4.3). Putting (3.3.9) in (3.3.8) and writing (3.3.8) in vector notations, we have

$$\begin{aligned}
\underline{y}_0 &= (\underline{\kappa}_0\tilde{\tilde{\mu}})^{-1}\underline{\kappa}_0[-R^{-1}(0)] \\
&= [\underline{\kappa}_0\underline{\mu}^*-\underline{\kappa}_0R^{-1}(0)\underline{e}]^{-1}[-\underline{\kappa}_0R^{-1}(0)] && \text{by (2.4.3)} \\
&= [\mu^{(1)}\underline{\kappa}_0\tilde{\underline{\mu}}^*-\underline{\kappa}_0R^{-1}(0)\underline{e}]^{-1}[-\underline{\kappa}_0R^{-1}(0)] && \text{by (2.3.15)} \\
&= [\mu^{(1)}-\underline{x}_0R^{-1}(0)\underline{e}]^{-1}[-\underline{x}_0R^{-1}(0)] && \text{by (3.2.12)} \\
&= -\xi^*\underline{x}_0R^{-1}(0) && \text{by (3.3.2).}
\end{aligned}$$

<u>Corollary 3.3.10</u>: The stationary probability that the queue is empty at an arbitrary epoch t is given by

$$\underline{y}_0\underline{e} = (1-\rho) \tag{3.3.11}$$

<u>Proof</u>: This follows easily from (3.3.7) and (3.3.2).

<u>Special Cases</u>:

1. For the M/G/1 queue (with group arrivals) it is easily verified that (3.3.7) reduces to $y_0=(1-\rho)$.
2. For the PH/G/1 queue (with single arrivals), formula (3.3.7) can be shown to reduce to

$$\underline{y}_0 = (\underline{\alpha}GT^{-1}\underline{e}-\mu^{(1)}\underline{\alpha}\tilde{\underline{\mu}})^{-1}\underline{\alpha}GT^{-1}$$

from which we get

$$\underline{y}_0\underline{e}=(-\underline{\alpha}GT^{-1}\underline{e}+\mu^{(1)}\underline{\alpha}\tilde{\underline{\mu}})^{-1}(-\underline{\alpha}GT^{-1}\underline{e}) \tag{3.3.12}$$

These are obtained by using (3.2.12) and the fact $R(0)=T$. In (3.3.12) it may be noted that $(-\underline{\alpha}GT^{-1}\underline{e})$ is the expected length of an idle period, for, a busy period starts with a phase given by $\underline{\alpha}$ whence the phase at the end of the busy period is given by $\underline{\alpha}G$, and further $-T^{-1}\underline{e}=\int_{0-}^{\infty} P(0,y)\underline{e}\,dy$. Thus $\underline{y}_0\underline{e}$ in (3.3.12) which is the stationary probability that the queue is empty at an arbitrary epoch t is simply the ratio of the expected duration of an idle period to the expected duration of the busy cycle, a result which is quite intuitive!

Let us now define

$$\underline{\delta}_0=\int_{0-}^{\infty} x \sum_{\nu=0}^{\infty} d\tilde{B}_\nu(x)\underline{e}=-R^{-1}(0)\underline{e}+\mu^{(1)}\underline{e}$$

and

$$\underline{\delta}_k=\int_{0-}^{\infty} x \sum_{\nu=0}^{\infty} d\tilde{A}_\nu(x)\underline{e}=\mu^{(1)}\underline{e}, \quad k\geq 1.$$

Also let $\underline{\delta}'=(\underline{\delta}_0',\underline{\delta}_1',\ldots.)$. Note that $\delta(i,j)$ is the mean sojourn time in (i,j) for the Markov Renewal Process $\tilde{Q}(\cdot)$. From this we have

<u>Theorem 3.3.13</u>: The mean recurrence time of (i,j) in the Markov Renewal Process $\tilde{Q}(\cdot)$ is given by

$$m^{-1}(i,j)=(\underline{x}\underline{\delta})^{-1}x(i,j)=\xi^{*}x(i,j) \qquad (3.3.14)$$

Proof: It suffices to verify that

$$\underline{x}\underline{\delta} = \underline{x}_0[-R^{-1}(0)\underline{e}+\mu^{(1)}\underline{e}]+\sum_{n=1}^{\infty}\underline{x}_n\mu^{(1)}\underline{e}$$

$$= -\underline{x}_0R^{-1}(0)\underline{e}+\mu^{(1)} \quad \text{since} \quad \sum_{n=0}^{\infty}\underline{x}_n\underline{e}=1$$

$$= (\xi^*)^{-1} \quad \text{by (3.3.2)}$$

We are now ready to compute the vectors $\underline{y}_i$, $i \geq 1$.

Theorem 3.3.15: For $i \geq 1$,

$$\underline{y}_i = \sum_{\nu=1}^{i} \xi^*[\underline{x}_0U_\nu(0)+\underline{x}_\nu]\int_{0-}^{\infty}\{1-\tilde{H}(u)\}P(i-\nu,u)du \tag{3.3.16}$$

Proof: By considering the epoch of the last departure before t we can write

$$P\{X(t)=i,J(t)=j|X(0)=i',J(0)=j'\} =$$

$$\int_{0-}^{t}\sum_{k=1}^{m} dR_{0k}^{i'j'}(u)\int_{0-}^{t-u}\{1-\tilde{H}(t-u-x)\}\sum_{\nu=1}^{i}\sum_{p=1}^{m}[d\tilde{U}_\nu(x)]_{kp}P_{pj}(i-\nu,t-u-x)$$

$$+\sum_{\nu=1}^{i}\int_{0-}^{t}\sum_{k=1}^{m} dR_{\nu k}^{i'j'}(u)P_{kj}(i-\nu,t-u)\{1-\tilde{H}(t-u)\}$$

Letting $t\to\infty$ and applying the Key Renewal Theorem (Theorem 6.3, p. 153, Çinlar [2]) we obtain

$y(i,j)=$

$$= \sum_{k=1}^{m} m^{-1}(0,k)\int_{0-}^{\infty}\left\{\int_{0-}^{t}\{1-\tilde{H}(t-x)\}\sum_{\nu=1}^{i}\sum_{p=1}^{m}[d\tilde{U}_\nu(x)]_{kp}P_{pj}(i-\nu,t-x)\right\}dt$$

$$+\sum_{\nu=1}^{i}\sum_{k=1}^{m} m^{-1}(\nu,k)\int_{0-}^{\infty}P_{kj}(i-\nu,t)\{1-\tilde{H}(t)\}dt$$

Using (3.3.14) and putting the above in matrix notations

$$\underline{y}_i = \xi^* \underline{x}_0 \int_{0-}^{\infty} \left\{ \int_{0-}^{t} \{1-\tilde{H}(t-x)\} \sum_{\nu=1}^{i} d\tilde{U}_\nu(x) P(i-\nu, t-x) \right\} dt$$

$$+\xi^* \sum_{\nu=1}^{i} \underline{x}_\nu \int_{0-}^{\infty} \{1-\tilde{H}(t)\} P(i-\nu, t) dt. \qquad (3.3.17)$$

Now

$$\int_{0-}^{\infty} \int_{0-}^{t} \{1-\tilde{H}(t-x)\} \sum_{\nu=1}^{i} d\tilde{U}_\nu(x) P(i-\nu, t-x) dt$$

$$= \int_{0-}^{\infty} \sum_{\nu=1}^{i} d\tilde{U}_\nu(x) \int_{x-}^{\infty} \{1-\tilde{H}(t-x)\} P(i-\nu, t-x) dt$$

$$= \sum_{\nu=1}^{i} U_\nu(0) \int_{0-}^{\infty} \{1-\tilde{H}(u)\} P(i-\nu, u) du.$$

Putting this in (3.3.17) and simplifying we get (3.3.16).

Theorem 3.3.18: The generating function $\underline{Y}(z) = \sum_{i=0}^{\infty} \underline{y}_i z^i$ is given by

$$\underline{Y}(z) = \begin{cases} \xi^*(z-1)\underline{X}(z)R^{-1}(z) & \text{if } 0 \leqq z < 1 \\ \underline{\theta} & \text{if } z=1 \end{cases} \qquad (3.3.19)$$

Proof: We have, for $0 \leqq z < 1$,

$$\underline{Y}(z) - \underline{y}_0 = \sum_{i=1}^{\infty} \sum_{\nu=1}^{i} \xi^*[\underline{x}_0 U_\nu(0) + \underline{x}_\nu] \int_{0-}^{\infty} \{1-\tilde{H}(u)\} P(i-\nu, u) du\, z^i$$

$$= \sum_{\nu=1}^{\infty} \xi^*[\underline{x}_0 U_\nu(0) + \underline{x}_\nu] z^\nu \int_{0-}^{\infty} \{1-\tilde{H}(u)\} \tilde{P}(z,u) du$$

$$= \xi^*[\underline{x}_0 \{U(z,0) - I\} + \underline{X}(z)] \int_{0-}^{\infty} \{1-\tilde{H}(u)\} \tilde{P}(z,u) du \qquad (3.3.20)$$

Under our assumption $R(0)\neq Q^*$ it is easily shown that $R(z)$ is stable for $0\leq z<1$ and thus $R^{-1}(z)$ exists for $0\leq z<1$. (The proof of this is exactly analogous to the one establishing the non-singularity of $R(0)$ presented in Section 1.3 and hence omitted.) Thus

$$\int_{0-}^{\infty} \{1-\tilde{H}(u)\}\tilde{P}(z,u)du=[A(z,0)-I]R^{-1}(z), \quad 0\leq z<1,$$

for,

$$A(z,0)=\int_{0-}^{\infty} \exp[R(z)t]d\tilde{H}(t).$$

Putting this in (3.3.20) and simplifying with (3.2.6) and (3.3.7) the expression for $\underline{Y}(z)$ for $0\leq z<1$ is obtained.

Now, by letting $z\to 1-$ in (3.3.20), we have

$$\underline{Y}(1-)=\xi^*\left[-\underline{x}_0R^{-1}(0)+\{-\underline{x}_0R^{-1}(0)Q^*+\underline{X}(1-)\}\int_{0-}^{\infty} \{1-\tilde{H}(u)\}\tilde{P}(1,u)du\right] \tag{3.3.21}$$

It is now easily shown that $\underline{Y}(1-)\underline{e}=1$. Now,

$$\{-\underline{x}_0R^{-1}(0)Q^*+\underline{X}(1-)\}\int_{0-}^{\infty} \{1-\tilde{H}(u)\}\tilde{P}(1,u)du$$

$$=\{-\underline{x}_0R^{-1}(0)Q^*+\underline{\theta}-\underline{x}_0R^{-1}(0)Q^*A(I-A+\Theta)^{-1}\}$$

$$\times \int_{0-}^{\infty} \{1-\tilde{H}(u)\}\exp(Q^*u)du \qquad \text{by (3.2.8)}$$

$$=\mu^{(1)}\underline{\theta}-\underline{x}_0R^{-1}(0)[I+A(I-A+\Theta)^{-1}]Q^*\int_{0-}^{\infty} \{1-\tilde{H}(u)\}\exp(Q^*u)du$$

using the commutativity of Q^* with A and Θ.

$$=\mu^{(1)}\underline{\theta}+\underline{x}_0R^{-1}(0)[I+A(I-A+\Theta)^{-1}][I-A]$$

Substituting this in (3.3.21) and post-multiplying by Q^* it is easily verified using $\underline{\theta}Q^*=\underline{0}$ that

$$\underline{Y}(1-)Q^* = \underline{0}.$$

We already noted that $\underline{Y}(1-)\underline{e}=1$. Now by the uniqueness of the invariant probability vector $\underline{Y}(1-)=\underline{\theta}$.

<u>Remark</u>: Note that the j-th component of $\underline{Y}(1-)$ is the stationary probability that the phase of the N-Process is j. Clearly this must be θ_j, for, $\underline{\theta}$ is the invariant probability vector of the Markov Process Q^* governing the phases.

We now verify the correctness of (3.3.19) by particularizing it to the

<u>Special Cases</u>:

<u>1. M/G/1 queue (with single arrivals)</u>: Noting that in this case $\xi^*=\lambda$ and $R(z)=\lambda(z-1)$, (3.3.19) reduces to $Y(z)=X(z)$. Also, after some tedious computations using (3.2.6) we can obtain

$$Y(z)=X(z)=(1-\rho)(1-z)H(\lambda-\lambda z)/\{H(\lambda-\lambda z)-z\}$$

<u>2. M/G/1 queue (with group arrivals)</u>: In this case after some tedious computations one gets

$$Y(z)=(1-\rho)H(\lambda-\lambda\phi(z))+\frac{\eta}{1-\phi(z)}X(z)\{1-H(\lambda-\lambda\phi(z))\},$$

for, in this case

$$y_0=(1-\rho),\ x_0=(1-\rho)/\eta,\ U(z,0)=\phi(z),\ -R^{-1}(0)=1/\lambda$$

where $\phi(z)$ is the p.g.f. of the group size and $\eta=\phi'(1-)$. The expression for $Y(z)$ given here coincides with that obtained earlier by Neuts [19].

CHAPTER IV

MOMENTS OF THE QUEUE LENGTH

4.1 Introduction

It was pointed out earlier that the first two moments of the queue length can be used to truncate the infinite system of equations

$$\underline{x}\tilde{Q}(\infty)=\underline{x}, \quad \underline{x}\underline{e}=1$$

defining the stationary probability vector $\underline{x}$. While it would be ideal to have efficient methods for optimally truncating the system above, nevertheless, the problem of obtaining suitable criteria for this purpose appear quite intractable, and in the absence of such methods one has to rely on some simple procedures such as using a "μ+3σ limit". Fortunately, computational experience reported by Neuts [18] and Lucantoni [11] seem to favor such a procedure. We shall, in this Chapter, obtain the moments of the stationary queue length distributions obtained in Chapter III.

The moments of the queue length are closely related to the derivatives of the Perron-Frobenius eigenvalue and associated eigenvectors of the matrix $A(z,0+)$ as $z \to 1-$. We derive the recurrence relations for computing these derivatives in Section 4.2 using which formulas for the first two moments of the queue length are derived in Section 4.3. For a semi-Markov process $\tilde{Q}(\cdot)$ having a structure more general than the one given by (1.4.1), a program for computing the vector $\underline{x}$ and its first two moments using similar techniques has been written in APL by David Lucantoni [11] and is seen to be very efficient. We refer the reader to [11] for details of this program.

4.2 Derivatives of the Perron-Frobenius Eigenvalue

Just as in Theorem 2.2.8 which establishes the irreducibility of A, we can show that the nonnegative matrix $A(z,0)$ for $0<z\leq 1$ is irreducible. We let $\eta(z)$ denote the uniquely defined Perron-Frobenius eigenvalue of $A(z,0)$ which is analytic for $z<1$. Let $\underline{u}(z)$ and $\underline{v}(z)$ be the right and left eigenvectors respectively of $A(z,0)$ corresponding to the eigenvalue $\eta(z)$, whose components are defined *to be analytic* for $z<1$ and such that

$$\underline{v}(z)\underline{u}(z)=\underline{v}(z)\underline{e}=1 \tag{4.2.1}$$

$$\underline{v}(1-)=\underline{\theta},\ \underline{u}(1-)=\underline{e} \tag{4.2.2}$$

hold, in addition to the defining equations

$$[A(z,0)-\eta(z)I]\underline{u}(z)=\underline{0} \tag{4.2.3}$$

$$\underline{v}(z)[A(z,0)-\eta(z)I]=\underline{0}. \tag{4.2.4}$$

Below we present a theorem which shows how the derivatives $\eta^{(n)}(1-)$, $\underline{u}^{(n)}(1-)$ and $\underline{v}^{(n)}(1-)$ can be evaluated recursively on n, provided that moments of a sufficiently large order exist for the entries of $A(z,0)$. Below we denote

$\left.\frac{\partial^n}{\partial z^n} A(z,0)\right|_{z=1-}$ by $A^{(n)}(1,0)$.

Theorem 4.2.5: The triples $\eta^{(n)}(1-)$, $\underline{u}^{(n)}(1-)$ and $\underline{v}^{(n)}(1-)$, $n\geq 0$ may be computed recursively for each n for which the matrix $A^{(n)}(1,0)$ is finite. The recursion formulas are

$$\eta^{(0)}(1-)=1,\ \underline{u}^{(0)}(1-)=\underline{e},\ \underline{v}^{(0)}(1-)=\underline{\theta} \tag{4.2.6}$$

$$\eta^{(1)}(1-)=\rho,\ \underline{u}^{(1)}(1-)=(I-A+\Theta)^{-1}\underline{\beta}-\rho\underline{e},$$
$$\underline{v}^{(1)}(1-)=\underline{\theta}A^{(1)}(1,0)(I-A+\Theta)^{-1}-\rho\underline{\theta} \tag{4.2.7}$$

and for $n\geq 2$,

$$\eta^{(n)}(1-)=\sum_{\nu=1}^{n}\binom{n}{\nu}\underline{\theta}A^{(\nu)}(1,0)\underline{u}^{(n-\nu)}(1-)-\sum_{\nu=1}^{n-1}\eta^{(\nu)}(1-)\underline{\theta}\underline{u}^{(n-\nu)}(1-) \tag{4.2.8}$$

$$\underline{v}^{(n)}(1-)=\sum_{\nu=0}^{n-1}\binom{n}{\nu}\underline{v}^{(\nu)}(1-)\{A^{(n-\nu)}(1,0)-\eta^{(n-\nu)}(1-)I\}(I-A+\Theta)^{-1} \tag{4.2.9}$$

$$\underline{u}^{(n)}(1-)=(I-A+\Theta)^{-1}\left[\sum_{\nu=1}^{n}\binom{n}{\nu}\{A^{(\nu)}(1,0)-\eta^{(\nu)}(1-)I\}\underline{u}^{(n-\nu)}(1-)\right]$$
$$-\sum_{\nu=1}^{n-1}\binom{n}{\nu}\underline{v}^{(\nu)}(1-)\underline{u}^{(n-\nu)}(1-)\underline{e} \tag{4.2.10}$$

Proof: Differentiating (4.2.3) n times with respect to z we get

$$\sum_{\nu=0}^{n}\binom{n}{\nu}[A^{(\nu)}(z,0)-\eta^{(\nu)}(z)I]\underline{u}^{(n-\nu)}(z)=\underline{0}.$$

Pre-multiplying this equation by $\underline{v}(z)$ and letting $z\to 1-$ yields in view of (4.2.2),

$$\sum_{\nu=0}^{n}\binom{n}{\nu}[\underline{\theta}A^{(\nu)}(1,0)-\eta^{(\nu)}(1-)\underline{\theta}]\underline{u}^{(n-\nu)}(1-)=\underline{0}$$

from which we have

$$\eta^{(n)}(1-)=\sum_{\nu=1}^{n}\binom{n}{\nu}\underline{\theta}A^{(\nu)}(1,0)\underline{u}^{(n-\nu)}(1-)-\sum_{\nu=1}^{n-1}\eta^{(\nu)}(1-)\underline{\theta}\underline{u}^{(n-\nu)}(1-) \tag{4.2.11}$$

which is finite if $A^{(n)}(1,0)$ is finite. In the case n=1 the second term in (4.2.11) is zero and the first term equals $\underline{\theta}A^{(1)}(1,0)\underline{e}=\underline{\theta}\underline{\beta}=\rho$ showing $\eta^{(1)}(1-)=\rho$.

Differentiating (4.2.4) once and letting $z\to 1-$ we get

$$\underline{v}^{(1)}(1-)[I-A]=\underline{\theta}[A^{(1)}(1,0)-\eta^{(1)}(1-)I]$$

or

$$\begin{aligned}\underline{v}^{(1)}(1-)&=\underline{\theta}[A^{(1)}(1,0)-\eta^{(1)}(1-)I](I-A+\Theta)^{-1}+\underline{v}^{(1)}(1-)\Theta\\&=\underline{\theta}A^{(1)}(1,0)(I-A+\Theta)^{-1}-\rho\underline{\theta},\end{aligned}$$

for, the second equality in (4.2.1) implies $\underline{v}^{(1)}(1-)\underline{e}=0$ whence $\underline{v}^{(1)}(1-)\Theta=\underline{0}$.

In general, differentiating (4.2.4) n times yields after letting $z\to 1-$,

$$\underline{v}^{(n)}(1-)[I-A]=\sum_{\nu=0}^{n-1}\binom{n}{\nu}\underline{v}^{(\nu)}(1-)[A^{(n-\nu)}(1,0)-\eta^{(n-\nu)}(1-)I].$$

Adding $\underline{v}^{(n)}(1-)\Theta$ to both sides and noting that $\underline{v}^{(n)}(1-)\Theta=\underline{0}$ because of the second equality in (4.2.1), we get (4.2.9).

Differentiating (4.2.3) n times we get after letting $z\to 1-$,

$$(I-A)\underline{u}^{(n)}(1-)=\sum_{\nu=1}^{n}\binom{n}{\nu}[A^{(\nu)}(1,0)-\eta^{(\nu)}(1-)I]\underline{u}^{(n-\nu)}(1-).$$

Adding $\Theta\underline{u}^{(n)}(1-)$ to both sides we can rewrite the above equation as

$$\underline{u}^{(n)}(1-)=(I-A+\Theta)^{-1}\sum_{\nu=1}^{n}\binom{n}{\nu}[A^{(\nu)}(1,0)-\eta^{(\nu)}(1-)I]\underline{u}^{(n-\nu)}(1-)$$
$$+[\Theta\underline{u}^{(n)}(1-)]\underline{e} \qquad (4.2.12)$$

Differentiating $\underline{v}(z)\underline{u}(z)\equiv 1$ n times and letting $z\to 1-$ we get

$$\Theta\underline{u}^{(n)}(1-)=-\sum_{\nu=1}^{n}\binom{n}{\nu}\underline{v}^{(\nu)}(1-)\underline{u}^{(n-\nu)}(1-)$$

$$=-\sum_{\nu=1}^{n-1}\binom{n}{\nu}\underline{v}^{(\nu)}(1-)\underline{u}^{(n-\nu)}(1-), \qquad (4.2.13)$$

for, $\underline{v}(z)\underline{e}\equiv 1$ implies $\underline{v}^{(n)}(1-)\underline{e}=0$. Using (4.2.13) in (4.2.12) we get (4.2.10). In the case n=1, the sum in (4.2.13) is zero and

$$\underline{u}^{(1)}(1-)=(I-A+\Theta)^{-1}[A^{(1)}(1,0)-\eta^{(1)}(1-)I]\underline{u}^{(0)}(1-)$$

$$=(I-A+\Theta)^{-1}[\underline{\beta}-\rho\underline{e}]=(I-A+\Theta)^{-1}\underline{\beta}-\rho\underline{e},$$

and the proof is complete.

4.3 Moments of the Queue Length

In this section we derive the first two moments of the queue length in terms of the derivatives obtained in Theorem 4.2.5. We wish to derive formulas for computing $\underline{X}'(1-)\underline{e}$ and $\underline{X}''(1-)\underline{e}$. To this end let us recall the equation (3.2.6)

$$\underline{X}(z)[zI-A(z,0)]=\underline{x}_0[U(z,0)-I]A(z,0)$$

Multiplying this by $\underline{u}(z)$ we get

$$[z-\eta(z)]\underline{X}(z)\underline{u}(z)=\eta(z)\underline{x}_0[U(z,0)-I]\underline{u}(z) \qquad (4.3.1)$$

Differentiating this with respect to z and rearranging

$$\underline{X}'(z)\underline{u}(z)=-\underline{X}(z)\underline{u}'(z)+$$

$$\frac{1}{z-\eta(z)}\Big[-(1-\eta'(z))\underline{X}(z)\underline{u}(z)+\eta'(z)\underline{x}_0\{U(z,0)-I\}\underline{u}(z)+$$

$$\eta(z)\underline{x}_0U'(z,0)\underline{u}(z)+\eta(z)\underline{x}_0\{U(z,0)-I\}\underline{u}'(z)\Big] \qquad (4.3.2)$$

Now, as $z\to 1-$,

$$-(1-\eta'(z))\underline{X}(z)\underline{u}(z)\to-(1-\rho)\underline{X}(1-)\underline{e}=-(1-\rho)$$

$$\eta'(z)\underline{x}_0\{U(z,0)-I\}\underline{u}(z)\to\rho\underline{x}_0\{U(1,0)-I\}\underline{e}=0$$

$$\eta(z)\underline{x}_0U'(z,0)\underline{u}(z)\to-\underline{x}_0R^{-1}(0)R'(1)\underline{e} \quad \text{using (2.2.5)}$$

and

$$\eta(z)\underline{x}_0\{U(z,0)-I\}\underline{u}'(z)\to$$

$$\underline{x}_0[-R^{-1}(0)R(1)][(I-A+\Theta)^{-1}\underline{\beta}-\rho\underline{e}] \quad \text{using (2.2.5) and (4.2.7)}$$

$$=-\underline{x}_0R^{-1}(0)(I-A+\Theta)^{-1}(T+T°A°)\underline{\beta}, \text{ for } R(1)=T+T°A° \text{ and } R(1)\underline{e}=\underline{0}.$$

$$=-\underline{x}_0R^{-1}(0)(I-A+\Theta)^{-1}[(A-I-\Theta)R'(1)\underline{e}+\Theta R'(1)\underline{e}] \text{ by (3.3.4)}$$

Putting all these together, it is seen that as $z\to 1-$ the term in square brackets in (4.3.2) converges to

$$-(1-\rho)-\underline{x}_0R^{-1}(0)\Theta R'(1)\underline{e}=0 \qquad (4.3.3)$$

by (3.3.5).

Thus to evaluate the limit of $\underline{X}'(z)\underline{u}(z)$ in (4.3.2) as $z\to 1-$, we apply L'Hospital's rule on the second term in the right side of (4.3.2). After some tedious computations it may be verified that this yields

$$\underline{X}'(1-)\underline{e}=-\underline{X}(1)\underline{u}'(1-)+$$

$$\frac{1}{2(1-\rho)}\Big[\eta''(1-)+2\rho\underline{x}_0U''(1,0)\underline{e}+2\rho\underline{x}_0\{U(1,0)-I\}\underline{u}'(1-)$$

$$+2\underline{x}_0U'(1,0)\underline{u}'(1-)+\underline{x}_0U''(1,0)\underline{e}+\underline{x}_0\{U(1,0)-I\}\underline{u}''(1-)\Big] \quad (4.3.4)$$

Now,

$$U(z,0)=I-R^{-1}(0)R(z)$$

implies

$$2\rho\underline{x}_0[U'(1,0)\underline{e}+\{U(1,0)-I\}\underline{u}'(1-)]=$$

$$-2\rho\underline{x}_0R^{-1}(0)R'(1)\underline{e}+2\rho\underline{x}_0\{-R^{-1}(0)(I-A+\Theta)^{-1}(T+T^{\circ}A^{\circ})\underline{\beta}\} \text{ by } (4.2.7)$$

$$=-2\rho\underline{x}_0R^{-1}(0)R'(1)\underline{e}+2\rho\underline{x}_0\{-R^{-1}(0)\}[-R'(1)\underline{e}+\Theta R'(1)\underline{e}] \text{ by } (3.3.4)$$

$$=-2\rho\underline{x}_0R^{-1}(0)\Theta R'(1)\underline{e}$$

$$=2\rho(1-\rho) \qquad \text{by } (3.3.5)$$

Using this in (4.3.4) we simplify the expression for $\underline{X}'(1-)\underline{e}$ and state the result as

Theorem 4.3.5: The stationary expected queue length immediately after a departure is given by

$$\underline{X}'(1-)\underline{e}=-\underline{X}(1-)\underline{u}'(1-)+\rho+$$

$$\frac{1}{2(1-\rho)}\Big[\eta''(1-)+2\underline{x}_0U'(1,0)\underline{u}'(1-)+\underline{x}_0U''(1,0)\underline{e}+$$

$$\underline{x}_0\{U(1,0)-I\}\underline{u}''(1-)\Big] \quad (4.3.6)$$

The correctness of (4.3.6) is verified by particularizing to the

Special Cases:

a) M/G/1 queue: Here $U(z,0)=z$, $\underline{u}(z)=1$. So, (4.3.6) reduces to $X'(1-)=\rho+\frac{1}{2(1-\rho)}\lambda^2\mu^{(2)}$, where λ is the arrival rate, and $\mu^{(2)}$ is the second (raw) moment of $\tilde{H}(\cdot)$.

b) M/G/1 queue with group arrivals: Here $U(z,0)=\phi(z)$, where $\phi(z)$ is the p.g.f. of the group size, and $u(z)=1$. So we can easily verify that

$$X'(1-)=\rho+\frac{1}{2(1-\rho)}[\eta''(1-)+x_0\phi''(1-)].$$

Now

$$\eta''(1-)=A''(1,0)=\lambda^2\mu^{(2)}\{\phi'(1-)\}^2+\frac{\rho}{\phi'(1-)}\phi''(1-),$$

since

$$A(z,0)=\int_{0-}^{\infty} e^{-\lambda[1-\phi(z)]t}\, d\tilde{H}(t).$$

It has already been shown that $x_0=(1-\rho)/\phi'(1-)$ whence we have

$$X'(1-)=\rho+\frac{1}{2(1-\rho)}\left[\lambda^2\mu^{(2)}\{\phi'(1-)\}^2+\frac{\phi''(1-)}{\phi'(1-)}\right]$$

Theorem 4.3.7:

$$\underline{X}'(1-)=[\underline{x}_0U'(1,0)A+\underline{x}_0\{U(1,0)-I\}A'(1,0)-\underline{X}(1)\{I-A'(1,0)\}](I-A+\Theta)^{-1}$$
$$+(\underline{X}'(1-)\underline{e})\underline{\theta}$$

Proof: Differentiating (4.3.1) with respect to z and letting $z\to 1-$ we get

$$\underline{X}'(1-)[I-A]=\underline{x}_0U'(1,0)A+\underline{x}_0\{U(1,0)-I\}A'(1,0)-\underline{X}(1)\{I-A'(1,0)\}.$$

Adding $\underline{X}'(1-)\Theta$ to both sides and simplifying we get the expression for $\underline{X}'(1-)$ given above.

We also obtain a formula for the stationary mean queue length in continuous time.

Theorem 4.3.8: The stationary mean queue length in continuous time is given by

$$\underline{Y}'(1-)\underline{e}=\rho\{\underline{X}'(1-)\underline{e}-\underline{x}_0R^{-1}(0)R'(1)\underline{e}\}+\frac{1}{2}\xi^{*2}\mu^{(2)}$$

$$-\xi^*\{-\underline{x}_0R^{-1}(0)Q^*+\underline{X}(1-)\}\{I-A+\mu^{(1)}Q^*\}(\tau^*\Theta-Q^*)^{-2}R'(1)\underline{e}, \tag{4.3.9}$$

where τ^* is any real number such that

$$\tau^*\geq\max_i(-Q^*_{ii}).$$

Proof: Differentiating (3.3.19) with respect to z, letting $z\to1-$ and post-multiplying by $\underline{e}$ we get

$$\underline{Y}'(1-)\underline{e}=\xi^*[\underline{x}_0U'(1,0)+\underline{X}'(1-)]\mu^{(1)}\underline{e}+$$

$$\xi^*[\underline{x}_0\{U(1,0)-I\}+\underline{X}(1-)]\int_{0-}^{\infty}\{1-\tilde{H}(t)\}\underline{\xi}(t)dt$$

$$=\rho\{\underline{X}'(1-)\underline{e}-\underline{x}_0R^{-1}(0)R'(1)\underline{e}\}+$$

$$\xi^*[-\underline{x}_0R^{-1}(0)Q^*+\underline{X}(1-)]\int_{0-}^{\infty}\{1-\tilde{H}(t)\}\Big[\Theta R'(1)\underline{e}t+$$

$$\{I-e^{Q^*t}\}\{\tau^*\Theta-Q^*\}^{-1}R'(1)\underline{e}\Big]dt \tag{4.3.10}$$

by (1.3.13)

where τ^* is any real number such that $\tau^*\geq\max_i(-Q^*_{ii})$

Now

$$\{I-e^{Q^*t}\}(\tau^*\Theta-Q^*)=\tau^*\Theta-\tau^*\Theta-Q^*\{I-e^{Q^*t}\},$$

for,

$$e^{Q^*t}\Theta=\Theta,$$

whence

$$\{I-e^{Q^*t}\}(\tau^*\Theta-Q^*)^{-1}=-Q^*(I-e^{Q^*t})(\tau^*\Theta-Q^*)^{-2}.$$

Thus

$$\int_{0-}^{\infty}\{1-\tilde{H}(t)\}\{I-e^{Q^*t}\}\{\tau^*\Theta-Q^*\}^{-1}R'(1)\underline{e}dt$$

$$=-\int_{0-}^{\infty}\{1-\tilde{H}(t)\}Q^*\{I-e^{Q^*t}\}dt\{\tau^*\Theta-Q^*\}^{-2}R'(1)\underline{e}$$

$$=-[\mu^{(1)}Q^*+(I-A)][\tau^*\Theta-Q^*]^{-2}R'(1)\underline{e}, \qquad (4.3.11)$$

for,

$$\int_{0-}^{\infty}\{1-\tilde{H}(t)\}Q^*e^{Q^*t}dt=A-I$$

as can be seen by integration by parts and (2.2.9).

Putting (4.3.11) in (4.3.10) we get (4.3.9).

We now verify the correctness of (4.3.9) by particularizing to the

Special cases:

a) M/G/1 queue: In this case (4.3.9) reduces to

$$Y'(1)=\rho\{(\rho+\frac{1}{2(1-\rho)}\lambda^2\mu^{(2)})+(1-\rho)\}+\frac{1}{2}(\frac{\rho}{\mu^{(1)}})^2\mu^{(2)}$$

$$=\rho+\frac{1}{2(1-\rho)}\lambda^2\mu^{(2)}=X'(1)$$

as is well-known.

b) M/G/1 queue with group arrivals: Here (4.3.9) reduces to

$$Y'(1)=\rho\{X'(1)-x_0(-\frac{1}{\lambda})\lambda\phi'(1-)\}+\frac{1}{2}(\frac{\rho}{\mu^{(1)}})^2\mu^{(2)}$$

$$=\rho+\frac{\rho}{2(1-\rho)}\left[\lambda^2\mu^{(2)}\{\phi(1-)\}^2+\frac{\phi''(1-)}{\{\phi'(1-)\}^2}\right]+\frac{1}{2}\{\lambda\phi'(1-)\}^2\mu^{(2)}$$

$$=\rho+\frac{1}{2(1-\rho)}\lambda^2\mu^{(2)}\{\phi'(1-)\}^2+\frac{1}{2(1-\rho)}\frac{\phi''(1-)}{\mu^{(1)}}-\frac{1}{2}\frac{\phi''(1-)}{\phi'(1-)}$$

$$=X'(1-)-\frac{1}{2}\frac{\phi''(1-)}{\phi'(1-)}$$

We are now ready to obtain the second (factorial) moment of $\underline{x}$. We state our result as

<u>Theorem 4.3.12</u>:

$$\underline{X}''(1-)\underline{e}=-2\underline{X}'(1-)\underline{u}'(1-)-\underline{X}(1-)\underline{u}''(1-)+\eta''(1-)-2\rho^2$$

$$+\frac{1}{3(1-\rho)}\Big[\eta'''(1-)+\{3\eta''(1-)+6\rho(1-\rho)\}\{\underline{X}'(1-)\underline{e}+\underline{X}(1)\underline{u}'(1-)\}$$

$$-3\rho\eta''(1-)+\underline{x}_0U'''(1-,0)\underline{e}+3\underline{x}_0U''(1-,0)\underline{u}'(1-)$$

$$+3\underline{x}_0U'(1-,0)\underline{u}''(1-)+\underline{x}_0\{U(1,0)-I\}\underline{u}'''(1-)\Big] \qquad (4.3.13)$$

<u>Proof</u>: Differentiating (4.3.2) twice with respect to z and rearranging the terms we get

$$\underline{X}''(z)\underline{u}(z)=-2\underline{X}'(z)\underline{u}(z)-\underline{X}(z)\underline{u}''(z)$$

$$+\frac{1}{(z-\eta(z))}\Big[-2\{1-\eta'(z)\}\underline{X}'(z)\underline{u}(z)-2\{1-\eta'(z)\}\underline{X}(z)u'(z)+$$

$$\eta''(z)\underline{X}(z)\underline{u}(z)+\eta''(z)\underline{x}_0\{U(z,0)-I\}\underline{u}(z)+2\eta'(z)\underline{x}_0U'(z,0)\underline{u}(z)$$

$$+2\eta'(z)\underline{x}_0\{U(z,0)-I\}\underline{u}'(z)+\eta(z)\underline{x}_0U''(z,0)\underline{u}(z)+$$

$$2\eta(z)\underline{x}_0U'(z,0)\underline{u}'(z)+\eta(z)\underline{x}_0\{U(z,0)-I\}\underline{u}''(z)\Big] \qquad (4.3.14)$$

Now as $z \to 1-$, the quantity in square brackets above converges to

$$-2(1-\rho)\underline{X}'(1-)\underline{e}-2(1-\rho)\underline{X}(1-)\underline{u}'(1-)+\eta''(1-)+2\rho\underline{x}_0U'(1,0)\underline{e}+$$
$$2\rho\underline{x}_0\{U(1,0)-I\}\underline{u}'(1-)+\underline{x}_0U''(1,0)\underline{e}+2\underline{x}_0U'(1,0)\underline{u}'(1-)+$$
$$\underline{x}_0\{U(1,0)-I\}\underline{u}''(1-)$$
$$=-2(1-\rho)\underline{X}'(1-)\underline{e}-2(1-\rho)\underline{X}(1-)\underline{u}'(1-)+\eta''(1-)+2\rho(1-\rho)$$
$$+\underline{x}_0U''(1,0)\underline{e}+2\underline{x}_0U'(1,0)\underline{u}'(1-)+\underline{x}_0\{U(1,0)-I\}\underline{u}''(1-)$$
$$=0 \tag{4.3.15}$$

where the last equality above is got by substituting the value of $\underline{X}'(1-)\underline{e}$ from (4.3.6) and the one before that is got by using (4.3.3) which implies

$$\underline{x}_0U'(1,0)\underline{e}+\underline{x}_0\{U(1,0)-I\}\underline{u}'(1-)=(1-\rho). \tag{4.3.16}$$

Thus to evaluate the limit as $z \to 1-$ of (4.3.14) we can apply L'Hospital's Rule on the third term in the right side of (4.3.14). After some gruesome computations we get

$$\underline{X}''(1-)\underline{e}=-2\underline{X}'(1-)\underline{u}'(1-)-\underline{X}(1-)\underline{u}''(1-)+$$
$$\frac{1}{3(1-\rho)}\Big[3\eta''(1-)\underline{X}'(1-)\underline{e}+3\eta''(1-)\underline{X}(1-)\underline{u}'(1-)+\eta'''(1-)$$
$$+3\eta''(1-)\underline{x}_0U'(1,0)\underline{e}+3\eta''(1-)\underline{x}_0\{U(1,0)-I\}\underline{u}'(1-)$$
$$+3\rho\underline{x}_0U''(1,0)\underline{e}+6\rho\underline{x}_0U'(1,0)\underline{u}'(1-)+3\rho\underline{x}_0\{U(1,0)-I\}\underline{u}''(1-)$$
$$+\underline{x}_0U'''(1,0)\underline{e}+3\underline{x}_0U''(1,0)\underline{u}'(1-)+3\underline{x}_0U'(1,0)\underline{u}''(1-)$$
$$+\underline{x}_0\{U(1,0)-I\}\underline{u}'''(1-)\Big]$$

$$=-2\underline{X}'(1-)\underline{u}'(1-)-\underline{X}(1-)\underline{u}''(1-)+$$

$$\frac{1}{3(1-\rho)}\Big[\eta'''(1-)+3\eta''(1-)\{\underline{X}'(1-)\underline{e}+\underline{X}(1)\underline{u}'(1-)+(1-\rho)\}$$

$$+3\rho\{2(1-\rho)\underline{X}'(1-)\underline{e}+2(1-\rho)\underline{X}(1-)\underline{u}'(1-)-\eta''(1-)-2\rho(1-\rho)\}$$

$$+\underline{x}_0U'''(1,0)\underline{e}+3\underline{x}_0U''(1,0)\underline{u}'(1-)+3\underline{x}_0U'(1,0)\underline{u}''(1-)$$

$$+\underline{x}_0\{U(1,0)-I\}\underline{u}'''(1-)\Big] \qquad \text{by using (4.3.16).}$$

Now (4.3.13) is got by using (4.3.15) in the above equation.

Remarks:

1. As pointed out earlier the formulas (4.3.6) and (4.3.13), in spite of their forbidding forms, are well-suited to numerical computations.

2. Higher moments of the queue length can, in principle, be found using similar techniques. But the resulting formulas become extremely difficult to implement. Usually, however, these are beyond the realm of practical interest.

CHAPTER V

THE VIRTUAL WAITING TIME

5.1 Introduction

The virtual waiting time $V(t)$ at time t is the length of time a customer who arrives at time t waits before entering service. Recall that $J(t)$ is the phase of the arrival process at $t+$. In this Chapter we derive the joint distribution of $V(t)$ and $J(t)$ as $t\to\infty$. To this end it shall be assumed that the arriving groups are served on a first-come-first-served basis; we shall not assume anything regarding the order of service within each group.

The formula for the steady-state Laplace-Stieltjes transform of $V(t)$ generalizes the well-known Pollaczek-Khinchin formula of the M/G/1 queue to the N/G/1 queue. It is also shown that the steady-state c.d.f. of $V(t)$ satisfies a Volterra system of integral equations. It is well-known that such a system can be solved numerically with considerable ease using classical methods.

5.2 Distribution of the Virtual Waiting Time

Let

$$\tilde{W}_j(x)=\lim_{t\to\infty} P[V(t)\leqq x, J(t)=j \mid X(0)=i, J(0)=j'],\quad x\geqq 0,\ 1\leqq j\leqq m$$

The Laplace-Stieltje's transform

$$W_j(s)=\int_{0-}^{\infty} e^{-sx}\, d\tilde{W}_j(x), \qquad \text{Re } s \geq 0.$$

We also let $\underline{\tilde{W}}(\cdot)$ and $\underline{W}(\cdot)$ denote the m-vectors whose j-th components are $\tilde{W}_j(\cdot)$ and $W_j(\cdot)$ respectively. Also $\tilde{H}^{(\nu)}$ will denote the ν-fold convolution of $\tilde{H}$ with itself.

Theorem 5.2.1:

$$\underline{W}(s)=\begin{cases} s\underline{y}_0\{sI+R[H(s)]\}^{-1} & \text{if } s>0 \\ \underline{\theta} & \text{if } s=0 \end{cases} \tag{5.2.2}$$

Remark: Formula (5.2.2) is a direct generalization of the Pollaczek-Khinchin formula to the N/G/1 model.

Proof: By a direct probabilistic argument considering the last departure epoch τ before t we obtain

$$P[0<V(t)\leq x, J(t)=j \mid X(0)=i, J(0)=j']=$$

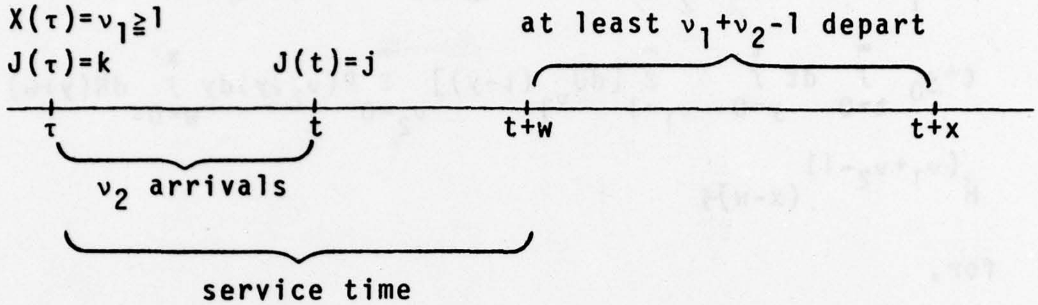
$$\sum_{\nu_1=1}^{\infty} \sum_{k=1}^{m} \int_{\tau=0-}^{t} dR^{ij'}_{\nu_1 k}(\tau) \sum_{\nu_2=0}^{\infty} P_{kj}(\nu_2, t-\tau) \int_{w=0-}^{x} d\tilde{H}(t+w-\tau)\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

$$+\sum_{k=1}^{m} \int_{\tau=0-}^{t} dR^{ij'}_{0k}(\tau) \int_{u=0-}^{t-\tau} \sum_{\nu_1=1}^{\infty} \sum_{p=1}^{m} [d\tilde{U}_{\nu_1}(u)]_{kp} \sum_{\nu_2=0}^{\infty} P_{pj}(\nu_2, t-\tau-u)\cdot$$

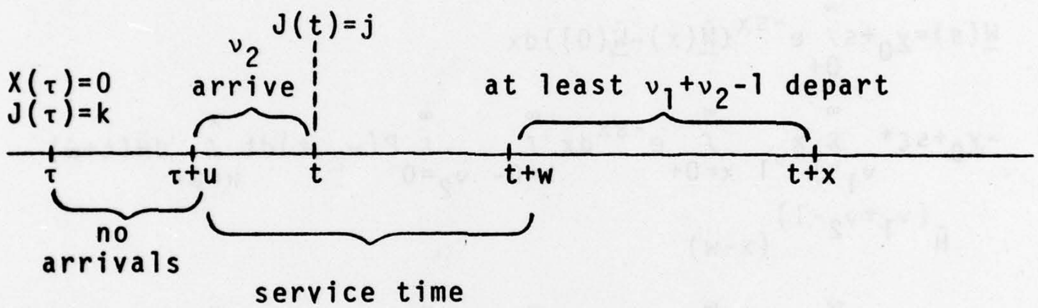
$$\int_{w=0-}^{x} d\tilde{H}(t+w-\tau-u)\, \tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

The two terms above on the right side correspond respectively to the two cases diagramatically shown below.

Case 1: Figure 1



Case 2: Figure 2



Letting $t\to\infty$ in the above equation and using the Key-Renewal Theorem (Theorem 6.3, p.153, Çinlar [2]), we get

$$\lim_{t\to\infty} P[0<V(t)\leq x, J(t)=j \mid X(0)=i, J(0)=j']$$

$$=\xi^* \sum_{\nu_1=1}^{\infty} \sum_{k=1}^{m} \int_{t=0-}^{\infty} x(\nu_1,k)dt \sum_{\nu_2=0}^{\infty} P_{kj}(\nu_2,t) \int_{w=0-}^{x} d\tilde{H}(t+w)\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

$$+\xi^* \sum_{k=1}^{m} \int_{t=0-}^{\infty} x(0,k)dt \int_{y=0-}^{t} \sum_{\nu_1=1}^{\infty} \sum_{p=1}^{m} [d\tilde{U}_{\nu_1}(t-y)]_{kp} \sum_{\nu_2=0}^{\infty} P_{pj}(\nu_2,y)dy$$

$$\int_{w=0-}^{x} d\tilde{H}(y+w)\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

whence,

$$\underline{\tilde{W}}(x)-\underline{y}_0=$$

$$\xi^*\sum_{\nu_1=1}^{\infty}\int_{t=0-}^{\infty}\underline{x}_{\nu_1}\sum_{\nu_2=0}^{\infty}P(\nu_2,t)dt\int_{w=0-}^{x}d\tilde{H}(t+w)\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)+$$

$$\xi^*\underline{x}_0\int_{t=0-}^{\infty}dt\int_{y=0-}^{t}\sum_{\nu_1=1}^{\infty}[dU_{\nu_1}(t-y)]\sum_{\nu_2=0}^{\infty}P(\nu_2,y)dy\int_{w=0-}^{x}d\tilde{H}(y+w)$$

$$\tilde{H}^{(\nu_1+\nu_2-1)}(x-w),$$

for,

$$\underline{\tilde{W}}(0)=\underline{y}_0.$$

Thus,

$$\underline{W}(s)=\underline{y}_0+s\int_{0+}^{\infty}e^{-sx}\{\underline{\tilde{W}}(x)-\underline{\tilde{W}}(0)\}dx$$

$$=\underline{y}_0+s\xi^*\sum_{\nu_1=1}^{\infty}\underline{x}_{\nu_1}\int_{x=0+}^{\infty}e^{-sx}dx\int_{t=0-}^{\infty}\sum_{\nu_2=0}^{\infty}P(\nu_2,t)dt\int_{w=0-}^{x}d\tilde{H}(t+w)$$

$$\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

$$+s\xi^*\underline{x}_0\sum_{\nu_1=1}^{\infty}U_{\nu_1}\int_{x=0+}^{\infty}e^{-sx}dx\int_{t=0-}^{\infty}\sum_{\nu_2=0}^{\infty}P(\nu_2,t)dt\int_{w=0-}^{x}d\tilde{H}(t+w)$$

$$\tilde{H}^{(\nu_1+\nu_2-1)}(x-w)$$

$$=\underline{y}_0+\xi^*\sum_{\nu_1=1}^{\infty}\underline{x}_{\nu_1}\int_{t=0-}^{\infty}\int_{w=0-}^{\infty}\sum_{\nu_2=0}^{\infty}P(\nu_2,t)dte^{-sw}d\tilde{H}(t+w)H^{\nu_1+\nu_2-1}(s)$$

$$+\xi^*\underline{x}_0\sum_{\nu_1=1}^{\infty}U_{\nu_1}\int_{t=0-}^{\infty}\int_{w=0-}^{\infty}\sum_{\nu_2=0}^{\infty}P(\nu_2,t)dte^{-sw}d\tilde{H}(t+w)H^{\nu_1+\nu_2-1}(s)$$

which on noting

$$\sum_{\nu=0}^{\infty}P(\nu,t)z^{\nu}=\exp\{R(z)t\}$$

yields

$$\underline{W}(s)=\underline{y}_0+\xi^*\sum_{\nu=1}^{\infty}\underline{x}_\nu\int_{t=0-}^{\infty}\int_{w=0-}^{\infty}\exp\{R[H(s)]t\}\cdot dt e^{-sw}d\tilde{H}(t+w)H^{\nu-1}(s)$$

$$+\xi^*\underline{x}_0\sum_{\nu=1}^{\infty}U_\nu\int_{t=0-}^{\infty}\int_{w=0-}^{\infty}\exp\{R[H(s)]t\}\cdot dt e^{-sw}d\tilde{H}(t+w)H^{\nu-1}(s) \tag{5.2.3}$$

Now,

$$\int_{0-}^{w}\exp\{(sI+R[H(s)])t\}\cdot dt=\{\exp(sI+R[H(s)])w-I\}\{sI+R[H(s)]\}^{-1}$$

Since the integral on the left side of the above equation is analytic in Re $s>0$, the right side has only removable singularities at a finite number of points in Re $s>0$, where the inverse fails to exist. Thus

$$\int_{t=0-}^{\infty}\exp\{R[H(s)]t\}dt\int_{w=0-}^{\infty}e^{-sw}d\tilde{H}(t+w)$$

$$=\int_{t=0-}^{\infty}\exp[\{sI+R[H(s)]\}t]\cdot dt\int_{y=t-}^{\infty}e^{-sy}d\tilde{H}(y)$$

$$=\left[\int_{w=0-}^{\infty}\exp\{R[H(s)]w\}\cdot d\tilde{H}(w)-H(s)I\right](sI+R[H(s)])^{-1}$$

Using this in (5.2.3), for $s>0$,

$$\underline{W}(s)=\underline{y}_0+\xi^*\left\{\underline{x}_0\sum_{\nu=1}^{\infty}U_\nu H^{\nu-1}(s)+\sum_{\nu=1}^{\infty}\underline{x}_\nu H^{\nu-1}(s)\right\}\cdot$$

$$\left[\int_{w=0-}^{\infty}\exp\{R[H(s)]w\}\cdot d\tilde{H}(w)-H(s)I\right](sI+R[H(s)])^{-1}$$

$$=\underline{y}_0+\xi^*\frac{1}{H(s)}[\underline{x}_0U(H(s),0)+\underline{X}(H(s))-\underline{x}_0]\cdot$$

$$[A(H(s),0)-H(s)I](sI+R[H(s)])^{-1}$$

$$=\underline{y}_0+\xi^*\underline{x}_0R^{-1}(0)R[H(s)](sI+R[H(s)])^{-1}$$

as can be seen by using (3.2.6) and (2.2.5). (5.2.2) now follows for $s>0$ from (3.3.7).

Multiplying both sides of (5.2.2) by $sI+R[H(s)]$,

$$\underline{W}(s)\{sI+R[H(s)]\}=s\underline{y}_0.$$

Letting $s\downarrow 0$, we get $\underline{W}(0+)Q^*=\underline{0}$. So to show that $\underline{W}(0+)=\underline{\theta}$ it now suffices to prove that $\underline{W}(0+)\underline{e}=1$. To this end multiply (5.2.3) by $\underline{e}$ and let $s\downarrow 0$ to get

$$\underline{W}(0+)\underline{e}=(\underline{y}_0\underline{e})+\lim_{s\downarrow 0} \xi^* \sum_{\nu=1}^{\infty} (\underline{x}_\nu\underline{e}) \int_{t=0-}^{\infty} dt \int_{w=0-}^{\infty} e^{-sw}d\tilde{H}(t+w)$$

$$+\lim_{s\downarrow 0} \xi^*\underline{x}_0\underline{e} \int_{t=0-}^{\infty} dt \int_{w=0-}^{\infty} e^{-sw}d\tilde{H}(t+w),$$

for,

$$\exp(Q^*t)\cdot\underline{e}\equiv\underline{e}$$

and

$$\sum_{\nu=1}^{\infty} U_\nu\underline{e}=\underline{e}.$$

Thus

$$\underline{W}(0+)\underline{e}=(1-\rho)+\xi^*\lim_{s\downarrow 0} \int_{t=0-}^{\infty} dt \int_{w=0-}^{\infty} e^{-sw}d\tilde{H}(t+w) \quad \text{by (3.3.11)}$$

$$=(1-\rho)+\xi^*\mu^{(1)}$$

$$=(1-\rho)+\rho=1,$$

and the proof is complete.

Remark: Note that for the M/G/1 queue, (5.2.2) reduces to the well-known Pollaczek-Khinchin Formula

$$W(s) = \frac{s(1-\rho)}{s-\lambda+\lambda H(s)}, \quad s>0,$$

for, in this case $y_0=(1-\rho)$ and $R(z)=-\lambda+\lambda z$.

<u>Theorem 5.2.4</u>: The vector $\underline{\tilde{W}}(x)$ satisfies the Volterra system of integral equations

$$\underline{\tilde{W}}(x)=\underline{y}_0+\underline{\tilde{W}}*K(x), \quad x\geqq 0 \tag{5.2.5}$$

where,

$$K(x)=\int_0^x\left[\Delta(\underline{\lambda})\Delta(\underline{e}-\sum_{k=0}^{\infty}\underline{p}(k)\tilde{H}^{(k)}(y))\right.$$
$$\left.-T\circ\sum_{k=0}^{\infty}q(k)\tilde{H}^{(k)}(y)-T^{\circ}A^{\circ}\circ\sum_{k=0}^{\infty}r(k)\tilde{H}^{(k)}(y)\right]dy, \quad x\geqq 0 \tag{5.2.6}$$

<u>Proof</u>: From (5.2.2) it is seen that for $s>0$,

$$\underline{W}(s)\{sI+R[H(s)]\}=s\underline{y}_0$$

or

$$\underline{W}(s)=\underline{y}_0+\underline{W}(s)\{-\frac{1}{s}R[H(s)]\}.$$

Now

$$-\frac{1}{s}R[H(s)]=\frac{1}{s}[\Delta(\underline{\lambda})\{I-\Delta(\underline{\phi}(H(s)))\}-T\circ\psi(H(s))-T^{\circ}A^{\circ}\circ\Phi(H(s))],$$

which shows that $-\frac{1}{s}R[H(s)]$ is the Laplace-Stieltjes transform of $K(\cdot)$ given in (5.2.6). Hence the Theorem.

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We discuss a single server queue is discussed whose input is the versatile Markovian point process, recently introduced by M.F. Neuts (c.f. Tech Report #77/13, Dept. of Statistics & CS, Univ. of Delaware), herein to be called the N-Process. Special cases of the N-Process ~~discussed earlier in the literature~~ include a number of complex models such as the Markov-modulated Poisson Process, the superposition of a Poisson Process and a Phase Type Renewal Process etc. This queueing → next page

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model has great appeal in its applicability to real world situations especially such as those involving inhibition or stimulation of arrivals by certain renewals. The paper presents formulas in forms which are computationally tractable and provides a unified treatment of many models which were discussed earlier by several authors and which turn out to be special cases. Among the topics discussed are busy period characteristics, queue length distributions, moments of the queue length and virtual waiting time. The analysis presented here serves as an example of the power of Markov Renewal Theory.